Math 580 Due Friday, August 29

1. The well ordering principle and mathematical induction.

We start by giving a precise description of the integers, $\mathbb{Z}$ which, unfortunately, will take a couple of pages. First we have two operations, addition, $+$, and multiplication, $\cdot$, (but we will usually write $x \cdot y$ as xy) which statisfy standard algebric rules ***commutativity***:

$$x + y = y + x, \qquad xy = yx,$$

associativity:

$$x + (y + z) = (x + y) + z, \qquad x(yz) = (xy)z,$$

and the ***distribution law***:

$$x(y + z) = xy + xz$$

which hold for all $x, y, z \in \mathbb{Z}$. There are two special elements 0 and 1 in $\mathbb{Z}$ with

$$0 \neq 1, \qquad x + 0 = x, \qquad x1 = x$$

for all $x \in Z$. The last purely algebric fact is that there are additive inverses. That is for all $x \in \mathbb{Z}$, there is a $y \in \mathbb{Z}$ such that

$$x + y = 0.$$

We will dinote this element y as $-x$ and write $x - y$ for $x + (-y)$.

The intgers have an ordering $<$ with the property that for all $x, y \in \mathbb{Z}$ exactly one of the following holds

$$x < y, \qquad x = 0, \qquad y < x.$$

This is the ***trichotomy law***. The ***transitive law***

$$x < y \quad \text{and} \quad y < z \qquad \implies x < z$$

holds for all $x, y, z \in Z$. Also

$$0 < 1.$$

For all $x, y, z \in \mathbb{Z}$

$$x < y \qquad \implies \qquad x + z < y + z.$$

Finally

$$x < y \quad \text{and} \quad z > 0 \qquad \implies \qquad xz < yz.$$

As usual we call the integers x with $x > 0$ ***positive*** and the ones with $x < 0$, ***negative***.

Proposition 1. *If $a, b, x, y \in \mathbb{Z}$ then*

$$a < b \quad \textit{and} \quad x < y \qquad \implies \qquad a + x < b + y.$$

In the future we will use this without giving a justification.

Problem 1. Prove this. *Hint:* Justify the following steps

$$\begin{aligned} a + x &< b + x \\ x + b &< y + z \\ b + x &< y + z \\ a + x &< b + y. \end{aligned}$$

□

Here another property that we have all used so often that we take it for granted.

Proposition 2. *If $x \neq 0$, then x is postive if and only if $-x$ is negative.*

Problem 2. Prove this. *Hint:* Assume toward a contradiction, that this is false. Then there is an $x \neq 0$ such that either x and $-x$ are both positive or x and $-x$ are both negative. In the first case $x > 0$ and $-x > 0$ and so by the last proposition $0 = x + (-x) > 0 + 0 = 0$ which is impossible. Do a similar calculation for the case they are both negative. □

Proposition 3. *For all $x \neq 0$, we have $x^2 > 0$.*

Proof. As $x \neq 0$, then either $x > 0$ or $x < 0$. In the first case multiply the inequality $x > 0$ by the positive number x to get

$$0 = x0 < xx = x^2.$$

In the second case $x < 0$, which implies $-x > 0$. We then multiply $-x > 0$ by the positive number $-x$ to get

$$0 = (-x)0 < (-x)(-x) = x^2.$$

□

Proposition 4. *For all $x, y, z \in \mathbb{Z}$*

$$z < 0 \quad \text{and} \quad x < y \qquad \implies \qquad xz > yz.$$

That is multiplication by a negative number reverses an inequality.

Problem 3. Prove this. *Hint:* $x < y$ implies $(y - x) > 0$ and $z < 0$ implies $-z > 0$. Therefore $(-z)(y - x) > 0$. □

We still have not given enough properties to uniquely single out the integer. For example the rational numbers statify all the above properties. To finish charaterizing them we need couple of definitions.

Definition 5. Let $S \subseteq \mathbb{Z}$ be a nonempty subset of the integers.

(a) S is ***bounded below*** iff there is some $x_0 \in \mathbb{Z}$ such that $x \geq x_0$ for all $x \in S$.
(b) S has a ***least element*** (also called a ***smallest element***) iff there a $y_0 \in S$ such that $y_0 \leq x$ for all $x \in S$. □

Well Ordering Principle. *Any nonempty subset of $\mathbb{Z}$ that is bounded below contains a least element.* □

While we will not prove it, it is true that the Well Ordering Property, together with the algebraic and order properties given above, characterizes the integers.

Theorem 6 (Principle of induction)**.** *Let $P \subseteq \mathbb{Z}$ be a subset of the integers such that*
(a) There is some $n_0 \in P$ (base case), and
(b) For all $n \geq n_0$ the implication

$$n \in S \qquad \Longrightarrow n + 1 \in P$$

holds (induction step). Then $n \in P$ for all $n \geq n_0$.

Proof. Assume, towards a contradiction, this is false. Then the set $S = \{n \geq n_0 : n \notin P\}$ is not empty. This set is clearly bounded below (by n_0). Therefore, by the Well Ordering Principle, the set S has a least element k_0. As $n_0 \in P$ we see $k_0 > n_0$. Therefore $k_0 - 1 \geq n_0$. As $k_0 - 1 < k_0$ and k_0 is the smallest element in S we have $k_0 - 1 \in P$. But then by condition (b) (the induction step) we have that $k_0 = (k_0 - 1) + 1 \in P$. Therefore we have the contradiction that $k_0 \notin P$ and $k_0 \in P$ both hold. $\square$

Let $\mathcal{P}(n)$ be some property of integers and assume that $\mathcal{P}(n_0)$ is true and that if for $n \geq n_0$

$$\mathcal{P}(n) \text{ is true} \qquad \Longrightarrow \qquad \mathcal{P}(n+1) \text{ is true.}$$

Then $\mathcal{P}(n)$ is true for all $n \geq n_0$. This follows directly from the last theorem by letting P be the set of integers

$$P = \{n : \mathcal{P}(n) \text{ is true.}\}.$$

Thus induction proofs split into two steps:

- **Base case:** First show that $\mathcal{P}(n_0)$ is true.
- **Induction step:** Assume $\mathcal{P}(n)$ is true (the *induction hypothesis*) and use it to show $\mathcal{P}(n+1)$ is true.

Example 7. Show that sum of the first n positive integers is n^2. That is

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1) = n^2.$$

In summation notation this is

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Base case: For $n = 1$ this becomes

$$1 = 1^2$$

which is certainly true. The next couple of cases, $n = 2$ and $n = 3$, give

$$1 + 3 = 2^2, \qquad 1 + 3 + 5 = 3^2.$$

(The cases $n = 2$ and $n = 3$ are not needed for the proof, but does give us confidence in the result.)

Induction step: Assume the result is true for n, that is the induction hypothesis

$$1^2 + 3^2 + 5^2 + \cdots + (2n - 1) = n^2.$$

holds. In the case of $n+1$ the sum on the left becomes $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 + (2(n+1)-1)$. That is we add $(2(n+1)-1) = 2n+1$. So add $(2(n+1)-1) = 2n+1$ to both sides of what we know to get

$$1^2+3^2+\cdots+(2n-1)+(2(n+1)-1) = n^2+(2(n+1)-1) = n^2+2n+1 = (n+1)^2.$$

Thus the result holds for $n+1$. This closes the induction and completes the proof. □

Proposition 8. *For $n \geq 0$ and $0 \leq k \leq n$ the bionimial coefficents*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are integers.

Problem 4. Use mathematical induction to prove this. *Hint:* First note $\binom{n}{0} = \binom{n}{n}$ so you only have to worry about the cases where $1 \leq k \leq n-1$. Use induction on n, the Pascal identity in the form

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and that the sum of two integers is an integer. □

Problem 5. Use induction to prove the binomial theorem. *Hint:* We wish to show that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds for all $n \geq 0$. We can also write this as

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

The main fact required is the Pascal identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Here is the induction step in going from $n = 4$ to $n = 5$.

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= (x+y)\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&= x\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&\quad + y\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&= \binom{4}{0}x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + \binom{4}{4}xy^4 \\
&\quad + \binom{4}{0}x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}xy^4 + \binom{4}{4}y^5 \\
&= \binom{4}{0}x^5 + \left[\binom{4}{0} + \binom{4}{1}\right]x^4y + \left[\binom{4}{1} + \binom{4}{2}\right]x^3y^2 \\
&\quad + \left[\binom{4}{2} + \binom{4}{3}\right]x^2y^3 + \left[\binom{4}{3} + \binom{4}{4}\right]xy^4 + \binom{4}{4}y^5 \\
&= \binom{4}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{4}{4}y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5
\end{aligned}$$

At the last step we have used that

$$\binom{4}{0} = \binom{4}{4} = \binom{5}{0} = \binom{5}{5} = 1.$$

Let us do the same calculation in summation notation.

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= (x+y)\sum_{k=0}^{4}\binom{4}{k}x^{4-k}y^k \\
&= x\sum_{k=0}^{4}\binom{4}{k}x^{5-k}y^k + y\sum_{k=0}^{4}\binom{4}{k}x^{4-k}y^k && \begin{pmatrix}\text{By induction}\\ \text{hypothesis}\end{pmatrix} \\
&= \sum_{k=0}^{4}\binom{4}{k}x^{5-k}y^k + \sum_{k=0}^{4}\binom{4}{k}x^{n-k}y^{k+1} && \begin{pmatrix}\text{Distribute } x\\ \text{into first sum}\\ \text{and } y \text{ into}\\ \text{second sum.}\end{pmatrix} \\
&= \sum_{k=0}^{4}\binom{4}{k}x^{5-k}y^k + \sum_{k=1}^{5}\binom{4}{k-1}x^{4-(k-1)}y^{(k-1)+1} && \begin{pmatrix}\text{Replace } k \text{ by}\\ k-1 \text{ in sec-}\\ \text{ond sum}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{4} \binom{4}{k} x^{5-k} y^k + \sum_{k=1}^{5} \binom{4}{k-1} x^{5-k} y^k && \text{(Simplify.)} \\
&= \binom{4}{0} x^5 + \sum_{k=1}^{4} \left(\binom{4}{k} + \binom{4}{k-1} \right) x^{5-k} y^k + \binom{4}{4} y^5 && \text{(Combine.)} \\
&= \binom{5}{0} x^5 + \sum_{k=1}^{4} \binom{5}{k} x^{5-k} y^k + \binom{5}{5} y^5 && \text{(Pascal identity.)} \\
&= \sum_{k=0}^{5} \binom{5}{k} x^{5-k} y^k
\end{aligned}
$$

which is the induction step between $n = 4$ and $n = 5$. □

Problem 6. A country has just 3¢ and 4¢ stamps. Use induction to show that it is possible to put exactly n¢s on a letter for all $n \geq 6$.

1.1. **Compete induction.**

Theorem 9 (Principle of compelte induction)**.** *$\mathcal{P}(n)$ be a property of integers. Assume*

(a) There is some n_0 such that $\mathcal{P}(n_0)$ is true, and
(b) For all $n \geq n_0$ the implication

$$\mathcal{P}(k) \text{ is true for all } k \text{ with } n_0 \leq k \leq n \quad \Longrightarrow \quad \mathcal{P}(n+1) \text{ is true.}$$

holds. Then $\mathcal{P}(n)$ is true for all $n \geq n_0$.

Proof. Let $\mathcal{Q}(n)$ be the statement that $\mathcal{P}(k)$ is true for $n_0 \leq k \leq n$. Then '$\mathcal{Q}(n_0)$' is true is is just that '$\mathcal{P}(n_0)$ is true'. And under the assumption (b) we have for $n \geq n_0$ that

$$\mathcal{Q}(n) \text{ is true.} \quad \Longrightarrow \quad \mathcal{Q}(n+1) \text{ is true.}$$

Thus by regular induction $\mathcal{Q}(n)$ is true for all $n \geq n_0$. But this implies $\mathcal{P}(n)$ is true for all $n \geq n_0$ as required. □

Often compete induction is called strong induction.

Here is an example of where compete induction is the right tool. We will see many more during the term. In this we will use that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

which we can see is true as it is a finite geometric series. Define a sequence by

$$a_0 = 1, \quad \text{and} \quad a_{n+1} = 1 + a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

If we compute the first several of these we get

$$\begin{aligned} a_1 &= 1 + a_0 = 2 \\ a_2 &= 1 + a_0 + a_1 = 4 \\ a_3 &= 1 + a_0 + a_1 + a_2 = 8 \\ a_4 &= 1 + a_0 + a_1 + a_2 + a_3 = 16, \\ a_5 &= 1 + a_0 + a_1 + a_2 + a_3 + a_4 = 32. \end{aligned}$$

At this point it is reasonable to conjecture that $a_n = 2^n$. If we try to prove this by usual induction we have that the base case holds (that is $n = 0$). The induction is that $a_n = 2^n$ and when we try to use in in the formula

$$a_{n+1} = 1 + a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n$$

we just get to replace a_n by n so that

$$a_{n+1} = 1 + a_0 + a_1 + a_2 + \cdots + a_{n-1} + 2^n.$$

Thus we are stuck as the induction hypothesis does not tell the values of a_k for $0 \le k \le n-1$. But if we are using strong induction, the induction step is that $a_k = 2^k$ for $0 \le k \le n$ (not just $a_n = 2^n$) and we can conclude

$$\begin{aligned} a_{n+1} &= 1 + a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n \\ &= 1 + 1 + 2 + 2^2 + \cdots + 2^{n-1} + 2^n \\ &= 1 + (2^{n+1} - 1) = 2^{n+1} \end{aligned}$$

and the proof is complete. $\square$

Problem 7. Define a sequence by

$$u_0 = 1, \quad \text{and} \quad u_{n+1} = 1 + 2(u_0 + u_1 + \cdots + u_{n-1} + u_n).$$

Compete the first five term of the sequence, make a conjecture about the value of n_n and use compete induction to prove your conjecture. $\square$

1.2. **Fibonacci Numbers.** The ***Fibonacci numbers*** are defined by the recursion

$$f_1 = 1, f_2 = 1, \quad \text{and for } n \ge 2 \quad f_{n+1} = f_n + f_{n-1}.$$

The first twenty are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

The equation $f_{n+1} = f_n + f_{n-1}$ implies each number is the sum of its two predecessors. Thus $21 = 8 + 13$, $144 = 89 + 55$ etc. These numbers have many surprising properties, and verifying some of these properties give good practice in doing proofs by induction.

Problem 8. Here are some identities for the Fibonacci numbers which you should prove by induction. In general the hard part of these results was guessing that they were true. Then it was not too hard to use induction to verify them.

(a) $f_1+f_2+f_3+\cdots+f_n = f_{n+2}-1$. (For example $1+1+2+3+5+8+13 = 34-1$.) *Hint:* The induction step going from $n=5$ to $n=6$ looks like this. We assume $f_1+f_2+f_3+f_4+f_5 = f_7-1$ Add f_6 to both sides to get $f_1+f_2+f_3+f_4+f_5+f_6 = f_7-1+f_6$ and use $f_6+f_7=f_8$ to conclude $f_7-1+f_6=f_8$.
(b) $f_1+f_3+f_5+\cdots+f_{2n-1} = f_{2n}$.
(c) $f_2+f_4+f_6+\cdots+f_{2n} = f_{2n+1}-1$.
(d) $f_1^2+f_2^2+f_3^2+\cdots+f_n^2 = f_nf_{n+1}$. □

Recreational Extra Credit Problem (An exact formula for f_n.). Prove ***Binet's formula*** (which is actually due to de Moivre) for f_n.

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$$

Hint: One way is to define b_n to be

$$b_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

and then do some messy algebra to show that $b_1 = b_2 = 1$ and $b_{n+1} = b_n + b_{n-1}$ for $n \geq 2$. Then $f_n = b_n$ follows by induction. □