Number Theory Homework.

In class we have proven that

Theorem 1. *If P is a lattice polygon and*

$$I(P) = \textit{Number of lattice points interior to } P$$
$$B(P) = \textit{Number of lattice points on boundary of } P.$$

then the area of P is

$$A(P) = I(P) + \frac{1}{2}B(P) - 1.$$

□

Problem 1. Show for any lattice polygon P that $2A(P)$ is an integer. □

Problem 2. Show that if P is a lattice polygon and $A(P)$ is an integer, then the number of lattice points on the boundary of P is even. □

The n-th ***Farey Series***, $\mathcal{F}_n$, is the set of rational numbers $r = \frac{p}{q}$ in lowest terms with $0 \le p \le q \le n$ and listed in increasing order. That is

$$\mathcal{F}_n = \left\{ \frac{p}{q} : \gcd(p,q) = 1, 0 \le \frac{p}{q} \le 1 \right\}$$

The first ten Farey series are

$$\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$
$$\mathcal{F}_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$
$$\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$
$$\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$
$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$
$$\mathcal{F}_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}$$
$$\mathcal{F}_7 = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1} \right\}$$
$$\mathcal{F}_8 = \left\{ \frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1} \right\}$$
$$\mathcal{F}_9 = \left\{ \frac{0}{1}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{1}{1} \right\}$$
$$\mathcal{F}_{10} = \left\{ \frac{0}{1}, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \right.$$
$$\left. \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{1}{1} \right\}$$

We proved the following in class.

Proposition 2. *The number of elements of $\mathcal{F}_n$ is*

$$\#\mathcal{F}_n = 1 + \sum_{k=1}^{n} \phi(k).$$

□

We also used Pick's Theorem to prove

Theorem 3 (Farey's Theorem)**.** *If*

$$\frac{a}{b} < \frac{a'}{b'}$$

are consecutive two terms in $\mathcal{F}_n$, then

$$a'b - ab' = 1.$$

□

Thus in

$$\mathcal{F}_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}$$

we have

$$1 = 1 \cdot 1 - 0 \cdot 6 = 1 \cdot 6 - 5 \cdot 1 = \cdots = 2 \cdot 5 - 3 \cdot 3 = 3 \cdot 3 - 2 \cdot 4 = 4 \cdot 4 - 3 \cdot 5 = \cdots$$

Problem 3. If

$$\frac{a}{b} < \frac{a'}{b'}$$

are consecutive terms in $\mathcal{F}_n$, then show $\gcd(a, a') = \gcd(b, b') = 1$. □

A consequence of Farey's Theorem is

Theorem 4 (Farey's Theorem form 2)**.** *If*

$$\frac{a}{b} < \frac{a'}{b'} < \frac{a''}{b''}$$

are consecutive terms in $\mathcal{F}_n$. Then

$$\frac{a'}{b'} = \frac{a + a''}{b + b''}.$$

In this it is not being claimed that $\dfrac{a + a''}{b + b''}$ is in lowest terms as written. For example in

$$\mathcal{F}_7 = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1} \right\}$$

we have

$$\frac{a}{b} = \frac{2}{7} < \frac{1}{3} = \frac{a'}{b'} < \frac{2}{5} = \frac{a''}{b''}$$

are consecutive and

$$\frac{a + a''}{b + b''} = \frac{2 + 2}{7 + 5} = \frac{4}{12} = \frac{1}{3} = \frac{a'}{b'}.$$

Problem 4. Prove the second form of Farey's Theorem. *Hint:* From the first form of Farey's Theorem we know $a'b - ab' = 1$ and $a''b' - a'b'' = 1$. We also know $0 = 1 - 1$. □