Sums of cubes and fourth powers.

Letting $x^{\underline{k}}$ be the falling factorial power

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1)$$

we have proven the following:

Proposition 1. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Proposition 2. *The equalities*

$$\begin{aligned}
x &= x^{\underline{1}} \\
x^2 &= x^{\underline{2}} + x^{\underline{1}} \\
x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\
x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\
x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

hold.

Proof. Messy calculations. □

Problem 1. Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3, \qquad \sum_{k=1}^{n} k^4$$

Solution: Starting with $\sum_{k=1}^{n} k^2$, use Propositions 2 and 1

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\
&= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} \\
&= (n+1)n\left(\frac{(n-1)}{3} + \frac{1}{2}\right) \\
&= (n+1)n\left(\frac{(2n+1)}{6}\right) \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

For $\sum_{k=1}^{n} k^3$ the same idea works

$$\begin{aligned}
\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} \left(k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}\right) \\
&= \sum_{k=1}^{n} \left(k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}\right) \\
&= \sum_{k=1}^{n} k^{\underline{3}} + 3\sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\
&= \frac{n^{\underline{4}}}{4} + 3\frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{n(n-1)(n-2)(n-3)}{4} + n(n-1)(n-2) + \frac{n(n-1)}{2} \\
&= \frac{n(n-1)(n-2)(n-3) + 4n(n-1)(n-2) + 2n(n-1)}{4} \\
&= \frac{n(n-1)\Big((n-2)(n-3) + 4(n-2) + 2\Big)}{4} \\
&= \frac{n(n-1)\left(n^2 - n\right)}{4} \\
&= \frac{n(n-1)n(n-1)}{4} \\
&= \frac{n^2(n-1)^2}{4}
\end{aligned}$$

Finally, and leaving out much of the algebra,

$$\begin{aligned}
\sum_{k=1}^{n} k^4 &= \sum_{k=0}^{n} (k^{\underline{4}} + 6k^{\underline{3}} + 7k^{\underline{2}} + k^{\underline{1}}) \\
&= \frac{n^{\underline{5}}}{5} + 6\frac{n^{\underline{4}}}{4} + 7\frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)}{5} + 6\frac{n(n-1)(n-2)(n-3)}{4} \\
&\qquad + 7\frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} \\
&= \frac{6n^5 + 15n^4 + 10n^3 - n}{30} \\
&= \frac{n(n+1)(2n+1)(2n^2 + 2n - 1)}{30}.
\end{aligned}$$