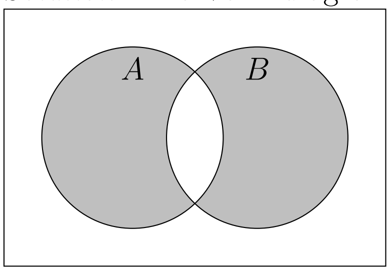
Math 574 **Test 1** Name: Answer Key

You are to use your own calculator, no sharing.
Show your work to get credit.

1. Use Venn diagrams to decide of the following are true for all sets A, B, and C.

(a) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. True or False? **True**

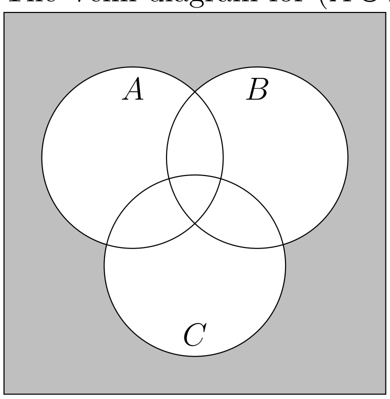
Solution: The Venn diagram for both is



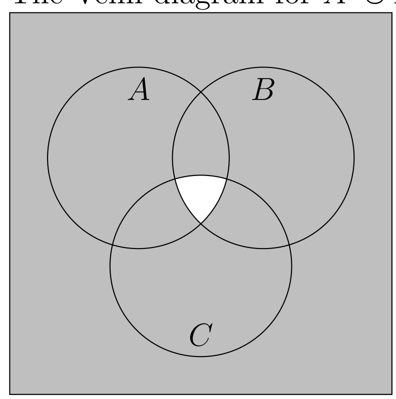
(b) $(A \cup B \cup C)^c = A^c \cup B^c \cup C^c$. True or False? **False**

Solution:

The Venn diagram for $(A \cup B \cup C)^c$ is



The Venn diagram for $A^c \cup B^c \cup C^c$ is



Anther way to see that they are not equal is to use DeMorgan's law: $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c \neq A^c \cup B^c \cup C^c$. $\square$

2. (a) Define what if means for a function $h \colon X \to Y$ to be ***surjective***.

Solution: For each $y \in Y$ there is a $x \in X$ with $h(y) = x$. $\square$

(b) Let $f \colon B \to C$ and $g \colon A \to B$ and assume that $f \circ g$ is surjective. Show that f is surjective.

Solution: We need to show that for any $c \in C$ there is a $b \in B$ with $f(b) = c$. We know that $f \circ g$ is surjective and therefore for any $c \in C$ there is an $a \in A$ with $f(g(a)) = c$. Set $b = g(a) \in B$. Then $b \in B$ and $f(b) = f(g(a)) = c$ which is just what we needed to show f is surjective. $\square$

3. (a) For a function $F(x)$ define the difference operator $\Delta F(x)$.

Solution: We assume that $F \colon \mathbb{Z} \to \mathbb{R}$ is a function from the integers to the real numbers. Then

$$\Delta F(x) = F(x+1) - F(x).$$

$\square$

(b) State the ***fundamental theorem of summation theory***.

Solution: Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a real valued function on the integers. Assume that F is an ***antidifference*** of f, that is $\Delta F = f$. Then for integers $a < b$

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

(This is not really much more than the standard result from calculus about telescoping series.) □

(c) Show that if $F(x) = \dfrac{-1}{x}$, then $\Delta F(x) = \dfrac{1}{x(x+1)}$.

Solution:

$$\begin{aligned}\Delta F(x) &= F(x+1) - F(x)\\ &= \frac{-1}{x+1} - \frac{-1}{x}\\ &= \frac{-x-(-x-1)}{x(x+1)}\\ &= \frac{1}{x(x+1)}\end{aligned}$$

□

(d) Find $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)}$. The sum is: $\dfrac{100}{101}$

Solution: Let $f(x) = \dfrac{1}{x(x+1)}$. Then by part (c) $F(x) = \dfrac{-1}{x}$ is an antidifference of $f(x)$. Therefore by the fundamental theorem of summation theory

$$\sum_{k=1}^{100} f(k) = F(101) - F(1)$$

that is

$$\sum_{k=1}^{100} \frac{1}{k(k+1)} = \frac{-1}{101} - \frac{-1}{1} = \frac{100}{101}.$$

□

4. A student puts one dollar in a box the first week of school. The next week he puts \$2.00 in the box, the (third) week after than he puts in \$4.00 the fourth week he puts in \$8.00, so that each week he puts in twice as much as the week before. (This is somewhat reworded from the ambiguous wording on the test that lead to two correct answers depending on the reading.)

(a) How much does he add to the box on the k^{th} week? 2^{k-1} dollars.

(b) What is the total in the box after 20 weeks? The total is 1,048,575

Solution: The total, S, will be the sum of the following geometric series

$$\begin{aligned}S &= 1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{19}\\ &= \frac{\text{first} - \text{next}}{1 - \text{ratio}}\\ &= \frac{1-2^{20}}{1-2}\\ &= 2^{20} - 1\\ &= 1{,}048{,}575\end{aligned}$$

$\square$

5. A motel has ten rooms that are in a row. A group of ten people is to be put into these rooms, one to a room.

(a) How many ways can this be done? Number of ways is: $\underline{10! = 3{,}628{,}800}$

Solution: This is just $P(10,10) = 10!$. $\square$

(b) How many ways can this be done if Alice and Bob have to have adjacent rooms?

Number of ways: $\underline{9 \cdot 2 \cdot 8! = 2 \cdot 9! = 725{,}760}$

Solution: There are 9 choices for the two adjacent rooms. Then Alice and Bob can be put into these two rooms in $2! = 2$ ways and then the remaining 8 people can be put into the remaining 8 rooms in $8!$ ways. This gives $9 \cdot 2 \cdot 8!$ ways of filling the rooms with Alice and Bob next to each other. $\square$

6. Give formulas for the following:

(a) $P(n,r) = n(n-1)\cdots(n-r+1) = n^{\underline{r}} = \dfrac{n!}{n-r)!}$.

(b) $P(n; n_1, n_2, n_3) = \dfrac{n!}{n_1! \cdot n_2! \cdot n_3!}$ where $n_1 + n_2 + n_3 = n$.

(c) $\dbinom{n}{r} = \dfrac{P(n,r)}{r!} = \dfrac{n(n-1)\cdots(n-r+1)}{r!} = \dfrac{n!}{r!(n-r)!}$.

7. (a) How many distinct strings can be made from the letters $a, a, b, b, b, c, c, c, c$?

The number is $P(9; 2, 3, 4) = \dfrac{9!}{2! \cdot 3! \cdot 4!} = 1{,}260$

(b) How many strings can be made from these letters if all the b's are adjacent?

The number is $P(7; 2, 1, 4) = \dfrac{7!}{2! \cdot 1! \cdot 4!} = \dfrac{7 \cdot 6!}{4! \cdot 2!} = 105$

Solution: One way to think of this is to view the string of three b's as one big letter letter $B = bbb$. Then the problem becomes how many strings can be made from a, a, B, c, c, c, c. There are $P(7; 2, 1, 4)$ such strings. $\square$

8. Consider the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) How many size 5 subsets does this set have? The number is $\dbinom{9}{5} = 126$

Solution: By definition $\dbinom{9}{5}$ is the number of size 5 subsets of the 9 element set X. And we have the formula

$$\binom{9}{5} = \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126.$$

$\square$

(b) How many subsets does X that contain 2 even digits and 3 odd digits?

The number is $\underline{60}$

Solution: There are $\binom{4}{2} = 6$ ways to choose the two even digits and $\binom{5}{3} = 10$ ways to choose the odd digits. This gives $6 \cdot 10 = 60$ ways to choose the required set. $\square$

9. Explain why for positive integers n and r with $r \leq n$ that $\dfrac{P(n,r)}{r!}$ is an integer. *Hint:* Recall that one of our formulas for $\binom{n}{r}$ is $\dfrac{P(n,r)}{r!}$.

Solution: Let X be any set of size n and $\mathcal{R}$ the set of all r element subsets of X. Then the number of elements, $\#(\mathcal{R})$, in $\mathcal{R}$ is an integer as it is the number of elements of a finite set. But we know that $\#(\mathcal{R}) = \dfrac{P(n,r)}{r!}$ and therefore $\dfrac{P(n,r)}{r!}$ is an integer. $\square$