Math 574 **Test 2** Name:____________________

You are to use your own calculator, no sharing.
Show your work to get credit.

1. Give formulas for the following:

(a) $P(n,r) = n^{\underline{r}} = n(n-1)(n-2)\cdots(n-r+1)$ (The number of factors in the product is n.) ☐

(b) $\binom{n}{k} = \frac{n^{\underline{r}}}{r!} = \frac{n!}{r!(n-r)!}$ ☐

(c) $\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$ ☐

2. Let X and Y be finite sets with $\#(X) = r$ and $\#(Y) = n$. Give formulas for the following

(a) The number of functions form X to Y.

Solution: For each $x \in X$ we can choose the value of $f(x)$ to be any $y \in Y$. Thus the number of functions is n^r. ☐

(b) The number of injections from X to Y.

Solution: Let $X = \{x_1, x_2, \ldots, x_r$. Then we can choose $f(x_1)$ to be any element of Y and there are n choices for this. Then we can choose $f(x_2)$ to be any element of Y other than $fx_1)$ and thus there are $n-1$ choices for $f(x_2)$. Then $f(x_3)$ can be chosen to be any element of Y other than $f(x_1)$ or $f(x_2)$. Thus $f(x_3)$ can be chosen in $n-2$ ways. Continuing like this we see that $f(x_j)$ can be chosen in $j - j + 1$ ways and therefore the number of injections from X to Y is

$$n(n-1)\cdots(n-r+1)$$

which is the same as $P(n,r)$. ☐

3. A barn on Animal Farm contains 10 cows and 12 sheep. The pigs running the farm want a committee from this barn to have 7 members, with 3 cows and 4 sheep. How many ways can this be done?

Solution: This is

$$(\text{Number of ways to choose cows}) \cdot (\text{Number of ways to choose sheep}) = \binom{10}{3} \cdot \binom{12}{4} = 59{,}400.$$

☐

4. Let X be a set with 11 elements.

(a) How many subsets does X have?

Solution: $2^{11} = 2{,}048$ subsets. ☐

(b) How many subsets of size 6 does X have?

Solution: $\binom{11}{6} = 462$. ☐

5. On Halloween a wicked witch wishes to pass out 19 identical poison apples to 4 princesses so that each princess gets at least 3 apples. How many ways can the witch do this?

Solution: Let x_j be the number of apples that j-th princess receives for $j = 1, 2, 3, 4$. Then we wish to find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 19 \qquad \text{with} \qquad x_1, x_2, x_3, x_4 \geq 3.$$

If we let $y_j = x_j - 3$, then $y_j \geq 0$ and

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 &= (x_1 - 3) + (x_2 - 3) + (x_3 - 3) + (x_4 - 3) \\ &= x_1 + x_2 + x_3 + x_4 - 12 \\ &= 19 - 12 \\ &= 7. \end{aligned}$$

So our problem is equivalent to finding the number of solutions to $y_1+y_2+y_3+y_4 = 7$ in nonnegative integers. But this is a type of problem we have done many times and the number is

$$\binom{7+3}{3} = \binom{10}{3} = 120.$$

□

6. How many seven card hands from a standard deck of 52 cards have a three of a kind and two pairs.

Solution: The number is

$$\frac{1}{2}\left(13\binom{4}{3}\right)\left(12\binom{4}{2}\right)\left(11\binom{4}{2}\right) = 123{,}552.$$

□

7. How many permutations of $a, b, c, \ldots, x, y, z$ contain at least one of the words "math", "is", "great".

Solution: To count the number that have "math" think of "math" as one letter. This leaves $24 - 4 = 22$ letters and so the resulting alphabet has $22 + 1 = 23$ letters with can be arranged in 23! ways. Similar considerations show that if

$$\begin{aligned} M &= \text{Permutations that contain "math".} \\ I &= \text{Permutations that contian "is"} \\ G &= \text{Permutations that contian "great"} \end{aligned}$$

then

$$\begin{aligned} \#(M) &= 23! \\ \#(I) &= 24! \\ \#(G) &= 22! \\ \#(M \cap I) &= 22! \\ \#(M \cap G) &= 0 && \text{(As "math" and "great" have letter in common)} \\ \#(I \cap G) &= 21! \\ \#(M \cap I \cap G) &= 0. \end{aligned}$$

The set of permutations that have at least one of our three words is $M \cup I \cup G$ and by the principle of inclusion and exclusion

$$\begin{aligned}\#(M \cup I \cup G) &= \#(M) + \#(I) + \#(G) - \#(M \cap I) - \#(M \cap G) - \#(I \cap G) + \#(M \cap I \cap G)\\ &= 23! + 24! + 22! - 22! - 0 - 21! + 0\\ &= 23! + 24! - 21!.\\ &= 646{,}249{,}327{,}529{,}952{,}706{,}560{,}000.\end{aligned}$$

□

8. (a) State the binomial theorem.
Solution: For any real numbers x and y and positive integer n the formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds. □

(b) Show that $\sum_{k=0}^{n} \binom{n}{k} 4^k (-1)^{n-k} = 3^n$.
Solution: By the binomial theorem we have

$$3^n = (4-1)^n = \sum_{k=0}^{n} \binom{n}{k} 4^k (-1)^{n-k}.$$

□

9. Find the number of ways that the letters from MISSISSIPPI can be arranged
(a) if there is no restriction on location of the letters.
Solution: There are 4 I's, 4 S's 2 P's and 1 M. So the number of arrangments is

$$\frac{11!}{4!4!2!1!} = 24{,}650.$$

□

(b) if all the I's stay together.
Solution: Then we can consider the I's as one letter, call it Ĩ. Then we have 4 M's 2 P's 1 Ĩ and 1 M. Thus the number is

$$\frac{8!}{4!2!1!1!} = 840.$$

□

10. What is the coefficient of $x^2y^3z^4$ in $(x - 2y + 3z)^9$.
Solution: It is

$$\frac{9!}{2!3!4!}(1)^2(-2)^3(3)^4 = -816{,}480$$

□

11. Find the number of solutions to $x + y + z + u + v = 20$ if x, y, z, u, v are
(a) nonnegative integers.
Solution: The number is

$$\binom{20+4}{4} = \binom{24}{4} = 10{,}626.$$

□

(b) positive integers.

Solution: Letting $\hat{x} = x - 1$, $\hat{y} = y - 1$ etc. We have that solving the original problem in positive integers is the same as solving

$$\hat{x} + \hat{y} + \hat{z} + \hat{u} + \hat{v} = 20 - 5 = 15$$

in nonnegative integers. So the number of solutions is

$$\binom{15+4}{4} = \binom{15}{4} = 1{,}365.$$

□

(c) nonnegative even integers.

Solution: The time let $x = 2\hat{x}$, $y = 2\hat{y}$ etc. Then the problem becomes

$$2\hat{x} + 2\hat{y} + 2\hat{z} + 2\hat{u} + 2\hat{v} = 20$$

that is

$$\hat{x} + \hat{y} + \hat{z} + \hat{u} + \hat{v} = 10$$

where now that variable are nonnegative integers. Thus the number of solutions is

$$\binom{10+4}{4} = \binom{14}{4} = 1001.$$

□

12. (10 points) Let $X = A \cup B$ where $\#(A) = m$, $\#(B) = n$ and $A \cap B = \varnothing$. Therefore $\#(X) = m + n$. If S is any subset of X with $\#(S) = r$, then for some k with $0 \le k \le r$ the set S will contain k elements from A and $r - k$ elements from B. This implies

$$(\#\text{ of size } r \text{ subsets of } X) = \sum_{k=0}^{r} (\#\text{ of size } k \text{ subsets of } A) \cdot (\#\text{ of size } r-k \text{ subsets of } B)$$

This implies an identity involving binomial coefficients. What is it?

Solution:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{n-k}.$$

□