

The Elements of the Calculus of Finite Difference

1. THE FUNDAMENTAL THEOREM OF SUMMATION THEORY.

Let $f: \mathbb{Z} \rightarrow \mathbb{R}$ be a function from the integers, \mathbb{Z} , to the real numbers, \mathbb{R} . We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^b f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^n f(k) = f(1) + f(2) + \cdots + f(n).$$

Definition 1. Let $f: \mathbb{Z} \rightarrow \mathbb{R}$. Then the ***difference***, Δf , of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1) + 2) - (3x + 2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax + b$	a
ar^x	$a(r-1)r^x$

Problem 1. Verify these. □

Theorem 1 (Fundamental Theorem of Summation Theory).

Let $f: \mathbb{Z} \rightarrow \mathbb{R}$ and let F be an ***anti-difference*** of f . That is $\Delta F = f$. Then for $a, b \in \mathbb{Z}$ with $a < b$

$$\sum_{k=a}^b f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^n f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}
 \sum_{k=a}^b f(k) &= \sum_{k=a}^b (F(k+1) - F(k)) \\
 &= \sum_{k=a}^b F(k+1) - \sum_{k=a}^b F(k) \\
 &= (F(a+1) + F(a+2) + \cdots + F(b) + F(b+1)) \\
 &\quad - (F(a) + F(a+1) + \cdots + F(b-1) + F(b)) \\
 &= F(b+1) - F(a)
 \end{aligned}$$

as required. \square

Theorem 1 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a , r and b are constants).

$f(x)$	$F(x)$
$ax + b$	$a \frac{x(x-1)}{2} + bx$
ar^x	$\frac{ar^x}{r-1}$

Problem 2. Verify these. (You just need to check $F(x+1) - F(x) = f(x)$). \square

Problem 3 (Sum of finite geometric series). Use that $\frac{ar^x}{r-1}$ is the anti-difference of ar^x and Theorem 1 to show

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1 - r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

\square

Problem 4. Sum the following geometric series.

- (a) $3 - 6 + 12 - 24 + 48 - 96 + 192$
- (b) $1 + x^2 + x^4 + \cdots + x^{20}$ for $x \neq 1$.
- (c) $1 + 1/2 + 1/2^2 + \cdots + 1/2^n$. In this case what is the limit as $n \rightarrow \infty$. \square

Problem 5. Here is an application of geometric series to finance. Assume that someone puts \$1,000 in the bank at the beginning of the year for 20 years. That is there will be 20 deposits. Assume that the bank pays 3% simple interest. The first deposit of \$1,000 will be worth $\$1,000(1.03)^{20}$ at the end of 20 years. The second deposit will be worth

\$1,000(1.03)¹⁹ at the end of the 20 years. What is the total amount in the bank at the end of the 20 years? \square

Problem 6. As a generalization of the last problem assume for n years we put a deposit of P_0 in the bank for n years (total of n deposits). Assume these earn 100r% simple interest each year. Find a formula for the total at the end of the n years. \square

2. FALLING FACTORIAL POWERS AND SUMS OF POWERS.

For Theorem 1 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 2. For natural number p define the *falling factorial power* of $x \in \mathbb{R}$ as $x^0 = 1$ and for $p \geq 1$

$$x^p = x(x-1)(x-2) \cdots (x-(p-1)).$$

(This product has p terms.) \square

For small values of p this becomes

$$x^0 = 1$$

$$x^1 = x$$

$$x^2 = x(x-1)$$

$$x^3 = x(x-1)(x-2)$$

$$x^4 = x(x-1)(x-2)(x-3)$$

$$x^5 = x(x-1)(x-2)(x-3)(x-4).$$

Proposition 1. If $f(x) = x^p$ where p is a natural number, then $\Delta f(x) = px^{p-1}$. That is

$$\Delta x^p = px^{p-1}.$$

Problem 7. Prove this. \square

Remark 1. The formula should remind you of the c formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. \square

Proposition 2. If $f(x) = x^p$ where p is a non-negative integer, then $F(x) = \frac{1}{p+1}x^{p+1}$ is an anti-difference of f .

Problem 8. Prove this as a corollary of Proposition 1 by noting (by replacing p by $p+1$), that $\Delta x^{p+1} = (p+1)x^p$ and dividing by $(p+1)$. \square

Problem 9. Show that if $p \geq 2$ that $1^p = 0$. (For example $1^3 = 1(1-1)(1-2) = 0$.) \square

Proposition 3. *If p is a positive integer, then*

$$\sum_{k=1}^n k^p = \frac{(n+1)^{p+1}}{p+1}.$$

Remark 2. This should remind you of the formula $\int_0^x t^p dt = \frac{x^{p+1}}{p+1}$.

Problem 10. Prove this. HINT: Let $f(x) = x^p$. Then $F(x) = \frac{x^{p+1}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 1

$$\sum_{k=1}^n f(k) = F(n+1) - F(1)$$

and use Problem 9 to see that $F(1) = 0$. \square

Proposition 4. *The equalities*

$$\begin{aligned} x &= x^1 \\ x^2 &= x^2 + x^1 \\ x^3 &= x^3 + 3x^2 + x^1 \\ x^4 &= x^4 + 6x^3 + 7x^2 + x^1 \\ x^5 &= x^5 + 10x^4 + 25x^3 + 15x^2 + x^1 \end{aligned}$$

hold.

Problem 11. Verify the first three of these. \square

Problem 12. Find formulas for

$$\sum_{k=1}^n k^2, \quad \sum_{k=1}^n k^3.$$

HINT: Here is the idea for $\sum_{k=1}^n k^2$. Using the last problem and Proposition 3

$$\begin{aligned}
 \sum_{k=1}^n k^2 &= \sum_{k=1}^n (k^2 + k^1) \\
 &= \sum_{k=1}^n k^2 + \sum_{k=1}^n k^1 \\
 &= \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} \\
 &= \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2}.
 \end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^n k^3$. □

3. A COUPLE OF TRIGONOMETRIC SUMS.

For your convenience we recall some trig identities:

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)$$

Problem 13. Let θ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show

$$\sin\left(\theta\left(x + \frac{1}{2}\right)\right) - \sin\left(\theta\left(x - \frac{1}{2}\right)\right) = 2 \sin\left(\frac{\theta}{2}\right) \cos(\theta x)$$

and therefore

$$F(x) = \frac{\sin\left(\theta\left(x - \frac{1}{2}\right)\right)}{2 \sin(\frac{\theta}{2})}.$$

is an anti-difference of

$$f(x) = \cos(\theta x).$$
□

Proposition 5. If $\sin(\frac{\theta}{2}) \neq 0$, then

$$\sum_{k=1}^n \cos(k\theta) = \frac{\sin\left((n + \frac{1}{2})\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2 \sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left((n + \frac{1}{2})\theta\right)}{2 \sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

Problem 14. Use Problem 13 and Theorem 1 to prove this. □

There is a similar formula for sums for the sine function.

Proposition 6. *If $\sin(\frac{\theta}{2}) \neq 0$, then*

$$\sum_{k=1}^n \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}.$$

Problem 15. Prove this. □