Mathematics 574 Homework

We have seen that the number of solutions to

$$x_1 + x_2 + \cdots + x_n = r$$

in nonnegative integers is $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$. As Max pointed out in class this implies that the number of solutions in nonnegative integers to

$$x_1 + x_2 + \cdots + x_n \le r$$

is

$$\sum_{k=0}^{r} \binom{n+k-1}{k}.$$

But we can count the number of solutions to $x_1 + x_2 + \cdots + x_n \le r$ in anther way. Add anther variable y and look for nonnegative solutions to

$$x_1 + x_2 + \cdots + x_n + y = r$$

Note that if $(x_1, \ldots, x_n, y)$ is a solution to this in nonnegative integers, then $(x_1, \ldots, x_n)$ is a solution to $x_1 + x_2 + \cdots + x_n \le r$ in nonnegative integers. Conversely if $(x_1, \ldots, x_n)$ is a solution to $x_1 + x_2 + \cdots + x_n \le r$ in nonnegative integers and we set $y = r - x_1 - x_2 - \cdots - x_n$ then $(x_1, \ldots, x_n, y)$ is a solution to $x_1 + x_2 + \cdots + x_n + y = r$ in nonnegative integers. Thus the two problems have the same number of solutions. Thinking of $y = x_{n+1}$ the number of solutions to $x_1 + x_2 + \cdots + x_n + x_{n+1} = r$ is

$$\binom{n+1+r-1}{r} = \binom{n+r}{r}.$$

1. Based on this explain why

$$\sum_{k=0}^{r} \binom{n+k-1}{k} = \binom{n+r}{r}.$$

□

2. Do the change of variable $n \mapsto n+1$ in the formula of the last problem to get

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+1+r}{r}.$$

This is Problem 1.53 from the text. □

3. Use the last problem to show

$$\sum_{k=0}^{r} \binom{n+k}{n} = \binom{n+1+r}{n+1}.$$

This is Problem 1.67 from the text. □