Mathematics 172

Quiz 5 Name: Key

You must show your work to get full credit.

1. Show that if $n \mid a$ and $n \mid b$ then $n \mid (5a - 13b)$.

Assume $n|a$ and $n|b$. Then

$a = kn$, $n = ln$

for some $k, l \in \mathbb{Z}$. Thus

$5a - 13b = 5kn - 13ln = (5k - 13l)n$.

As $5k - 13l \in \mathbb{Z}$ this implies $n \mid (5a - 13b)$

2. Show that if $5 \mid x^4$, then $5 \mid x$.

We prove the contrapositive: If $5 \nmid x$ then $5 \nmid x^4$.

Assume $5 \nmid x$. Then there are 4 cases

<u>case 1</u> $x \equiv 1 \mod 5$ so $x^4 \equiv 1^4 \equiv 1 \mod 5$ so $5 \nmid x^4$.

<u>case 2</u> $x \equiv 2 \mod 5$ so $x^4 \equiv 2^4 \equiv 16 \equiv 1 \mod 5$ so $5 \nmid x^4$.

<u>case 3</u> $x \equiv 3 \mod 5$ so $x^4 \equiv 3^4 \equiv 81 \equiv 1 \mod 5$ so $5 \nmid x^4$.

<u>case 4</u> $x \equiv 4 \mod 5$. so $x^4 \equiv 4^4 \equiv 256 \equiv 1 \mod 5$ so $5 \nmid x^4$.

Thus in all cases $5 \nmid x^4$.

3. Show $\sqrt[4]{5} = 5^{\frac{1}{4}}$ is irrational. Towards a contradiction assume $\sqrt[4]{5} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and this fraction in lowest terms. Then $a = \sqrt[4]{5}\, b$. so

$a^4 = 5b^4$.

This shows $5 \mid a^4$ and so (Prob. 2) $5 \mid a$. Therefore $a = 5k$ for some $k \in \mathbb{Z}$. Using this in $a^4 = 5b^4$ gives $(5k)^4 = 5b^4$. so

$b^4 = 5(5^2 k^4)$.

Thus $5 \mid b^4$. By Prob. 2 $5 \mid b$, so $b = 5l$ for some $l \in \mathbb{Z}$. But then $\frac{a}{b} = \frac{5k}{5l}$ is not in lowest terms, a contradiction

4. Show $\dfrac{\sqrt[4]{5}}{1+\sqrt[4]{5}}$ is irrational. Towards a contradiction assume

$r = \dfrac{\sqrt[4]{5}}{1+\sqrt[4]{5}}$ is rational. Then

$$r(1+\sqrt[4]{5}) = \sqrt[4]{5}$$

$$r\sqrt[4]{5} - \sqrt[4]{5} = -r$$

$$(r-1)\sqrt[4]{5} = -r$$

$$\sqrt[4]{5} = \frac{-r}{r-1}$$

which implies $\sqrt[4]{5}$ is rational, contradicting Problem 3.

5. Use that $3 \times 4 = 12 \equiv 1 \mod 11$ to show that $11 \mid 4a$ implies that $11 \mid a$.

Assume $11 \mid 4a$. Then

$$4a \equiv 0 \mod 11.$$

multiply by 3 to get

$$3 \cdot 4a \equiv 3 \cdot 0 \mod 11$$

$$12a \equiv 0 \mod 11$$

$$a \equiv 0 \mod 11$$

(as $12 \equiv 1 \mod 11$). This shows

$11 \mid a$.