Mathematics 300

Quiz 9

Name: Key

You must show your work to get full credit.

1. Show that if $(x+y)^3 = x^3 + y^3$, then $x = 0$ or $y = 0$ or $x + y = 0$.

Assume $(x+y)^3 = x^3 + y^3$.

Then $\cancel{x^3} + 3x^2y + 3xy^2 + \cancel{y^3} = \cancel{x^3} + \cancel{y^3}$

$$3x^2y + 3xy^2 = 0$$

$$3xy(x+y) = 0$$

so $x = 0$ or $y = 0$ or $x + y = 0$.

2. Prove that if $3 \nmid n$, then $n^2 \equiv 1 \mod 3$.

Assume $3 \nmid n$. Then there are two cases

<u>case 1</u> $n \equiv 1 \mod 3$.

Then $n^2 \equiv 1^2 \equiv 1 \mod 3$

<u>case 2</u> $n \equiv 2 \mod 3$.

Then $n^2 \equiv 2^2 \equiv 4 \equiv 1 \mod 3$.

3. Show that α is irrational if and only if $\dfrac{2\alpha+1}{\alpha}$ is irrational. There are two implications

(1) α irrational implies $\frac{2\alpha+1}{\alpha}$ irrational.

We show the contrapositive: If $r = \frac{2\alpha+1}{\alpha}$ is rational, then α is rational.

$$r\alpha = 2\alpha + 1$$

$$(r-2)\alpha = 1$$

$$\alpha = \frac{1}{r-2}$$ so α is rational.

(2) $\frac{2\alpha+1}{\alpha}$ irrational implies α irrational. The contrapositive is α rational implies $\frac{2\alpha+1}{\alpha}$ rational which is clearly true.

4. Show that if $n \mid a$ and $n \mid b$ and x and y are integers, then $n \mid (ax + by)$.

Assume $n|a$ and $n|b$. Then

$a = kn$ and $b = ln$ for some $k, l \in \mathbb{Z}$.

So $ax+by = knx + lny = (kx+ly)n$

and $kx+ly \in \mathbb{Z}$. Thus $n \mid (ax+by)$.

5. Give contrapositive proof that if x^5 is even, then x is even.

The contrapositive is x odd implies x^5 odd.

Assume x is odd. Then $x \equiv 1 \mod 2$.

So $x^5 \equiv 1^5 \equiv 1 \mod 2$ so x^5 is odd.

6. Prove that $\sqrt[5]{2}$ is irrational. Towards a contradiction assume that $\sqrt[5]{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, a/b in lowest terms. Then $a = \sqrt[5]{2}\, b$. Thus $a^5 = 2b^5$. Therefore a^5 is even, and by last problem a is even, say $a = 2k$ with $k \in \mathbb{Z}$. Use this in $a^5 = 2b^5$ to get $(2k)^5 = 2b^5$, that is $b^5 = 16a^5$. This implies b^5 is even and so (last problem again) b is even. That is $b = 2l$ for some $l \in \mathbb{Z}$. But then $\frac{a}{b} = \frac{2k}{2l}$ contradiction that a/b is in lowest terms.