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Mathematics 300 Test 2 Name:

You are to use your own calculator, no sharing.
Show your work to get credit.

1. (15 points) (a) If a and b are integers define what it means for a to ***divide*** b. (In symbols $a \mid b$.)

There exists an integer c where $ac=b$.

(b) Prove that if $a \mid b$ that $2a^2 \mid (6ab+4b^3)$.

Proof: Assume $a \mid b$.

So $ac=b$ for some $c \in \mathbb{Z}$.

Thus $6ab+4b^3 = 6a^2c+4(ac)^3 = 6a^2c+4a^3c^3 = 2a^2(3c+2c^3)$.

So $6ab+4b^3 = 2a^2(d)$ for some $d \in \mathbb{Z} \Rightarrow 3c+2c^3$. ∎

Therefore $2a^2 \mid (6ab+4b^3)$.

Good work

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2. (15 points) (a) Let n be a positive integer and a and b any integers. Define $a \equiv b \mod n$.

If $a \equiv b \mod n$, then $n \mid (a-b)$. This means that a and b have the same remainder when divided by n.

(b) Prove that if $a \equiv b \mod n$ and $b \equiv c \mod n$, then $a \equiv c \mod n$.

Proof: Assume $a \equiv b \mod n$ and $b \equiv c \mod n$.

So $n \mid (a-b)$ and $n \mid (b-c)$.

Thus $a-b=nd$ and $b-c=ne$ for some $d,e \in \mathbb{Z}$.

So $(a-b)+(b-c) = nd+ne = n(d+e)$.

Thus $a-c=nf$ for some $f \in \mathbb{Z} = d+e$.

So $n \mid (a-c)$, therefore $a \equiv c \mod n$.

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3. (20 points) (a) Define what it means for the real number r to be a *rational number*.

r is a rational number when $r = \frac{a}{b}$ where a, b are integers.

(b) Prove that if x is an integer and $3 \mid x^2$, then $3 \mid x$.

We will prove by contra~~diction~~positive $\Rightarrow$ If $3 \nmid x$ then $3 \nmid x^2$.

Assume $3 \nmid x$. There are two cases.

Case 1. $x \equiv 1 \bmod 3$

$x^2 \equiv (1)^2 \equiv 1 \bmod 3 \therefore 3 \nmid x^2$

Case 2. $x \equiv 2 \bmod 3$

$x^2 \equiv (2)^2 \equiv 2 \bmod 3 \therefore 3 \nmid x^2$

Therefore, in all cases, $3 \nmid x^2$.

(c) Prove that $\sqrt{3}$ is not a rational number.

Towards a contradiction, assume $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some integers a and b, $\frac{a}{b}$ is in lowest terms, and $b \neq 0$.

So $b\sqrt{3} = a$ $\quad 3b^2 = a^2$. As shown in part b, since $3 \mid a^2$, $3 \mid a$.

so $a = 3c$ for some integer c. So $3b^2 = (3c)^2$. Thus $3b^2 = 9c^2$

$b^2 = 3c^2$. Also as shown in part b, since $3 \mid b^2$, $3 \mid b$. So $b = 3d$ for some integer d. Therefore $\frac{a}{b} = \frac{3c}{3d}$. This shows that the fraction is not in lowest terms, a contradiction.

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good job

6. (10 points) Let x be an integer. Show that $x^3 + 6$ is even if and only if x is even.

Two Statements to prove.

Case 1: x^3+6 is even $\Rightarrow$ x is even.

proof by Contrapositive: x is odd $\Rightarrow$ x^3+6 is odd.

If x is odd, $x \equiv 1 \bmod 2$. So, $x^3+6 \equiv 1^3+6 \bmod 2$

$\equiv 7 \bmod 2 \equiv 1 \bmod 2$. So, x^3+6 is odd.

Case 2: x is even $\Rightarrow$ x^3+6 is even.

If x is even, $x \equiv 0 \bmod 2$. So, $x^3+6 \equiv 0^3+6 \bmod 2$

$\equiv 6 \bmod 2 \equiv 0 \bmod 2$. Thus x^3+6 is even.

Therefore, since both cases hold true,

x^3+6 is even if and only if x is even.

7. (10 points) Show that for any real number x, that $(x + 1)^3 = x^3 + 1^3$ if and only if $x = 0$ or $x = -1$.

Two cases to prove.

Case 1: $(x+1)^3 = x^3+1^3$ if $x=0$ or $x=-1$.

So, $\cancel{x^3}+3x^2+3x\cancel{+1}=\cancel{x^3}\cancel{+1}$ which means $3x^2+3x=0$.

Therefore $x(3x+3)=0$ means $x=0$, or $3x+3=0$, and

$3x=-3$, so $x=-1$. Thus $x=0$ or $x=-1$.

Case 2: If $x=0$ or $x=-1$, then $(x+1)^3 = x^3+1^3$.

For $x=0$, we have $(0+1)^3 = 0^3+1^3$. So, $1=1$.

For $x=-1$, we have $(-1+1)^3=(-1)^3+1^3$, which means

$0=-1+1$, and so $0=0$. Therefore, for $x=0$ or $x=-1$,

$(x+1)^3=x^3+1^3$.

Since both cases hold true, $(x+1)^3=x^3+1^3$ if and only if $x=0$ or $x=-1$.

Good

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8. (10 points) Show that for all integers x that $x^3 \equiv x \mod 3$.

Proof: Assume $x \in \mathbb{Z}$.

There are 3 cases, $x \equiv 0 \mod 3$, $x \equiv 1 \mod 3$, and $x \equiv 2 \mod 3$.

Case 1: $x \equiv 0 \mod 3$.

So $x^3 \equiv 0^3 = 0 \mod 3$.

Case 2: $x \equiv 1 \mod 3$.

So $x^3 \equiv 1^3 = 1 \mod 3$.

Case 3: $x \equiv 2 \mod 3$.

So $x^3 \equiv 2^3 = 8 \equiv 2 \mod 3$.

Therefore for all cases $x^3 \equiv x \mod 3$.

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