Mathematics 554H/701I Homework

1. From Page 12 the text do the following:

(a) Problem 2 (look up the definition of the Cartesian product $A \times B$ in the text.)

(b) Problems 4a,b (look up the definition of $A - B$ in the text and use Venn diagrams).

(c) Problem 5a.

1. Some useful algebra.

There are some algebraic identities we will need during the term. You recall that

$$x^2 - y^2 = (x - y)(x + y)$$

and may recall that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

These generalize. For any positive integer, n, and all real numbers x and y

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}).$$

Let us check this for $n = 4$. We start with the right side and simplify.

$$\begin{aligned}(x - y)(x^3 + x^2y + xy^2 + y^3) &= x(x^3 + x^2y + xy^2 + y^3) - y(x^3 + x^2y + xy^2 + y^3)\\ &= x^4 + x^3y + x^2y^2 + xy^3 - x^3y - x^2y^2 - xy^3 - y^4\\ &= x^4 - y^4\end{aligned}$$

2. Prove that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$$

for all positive integers n and all $x, y \in \mathbb{R}$. □

A related identity is that for all positive integers n and real numbers a, r with $r \neq 1$

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1 - r}.$$

Here is a proof when $n = 4$. Set

$$S = a + ar + ar^2 + ar^3 + ar^4.$$

Multiply by r

$$rS = ar + ar^2 + ar^3 + ar^4 + ar^5$$

Now subtract

$$\begin{aligned}(1 - r)S = S - rS =& a + ar + ar^2 + ar^3 + ar^4\\ &- ar - ar^2 - ar^3 - ar^4 - ar^5\\ =& a - ar^5.\end{aligned}$$

As $(1-r) \neq 0$ we can divide by $(1-r)$ to get

$$S = \frac{a - ar^5}{1-r}.$$

3. Prove that for any positive integer n and any real numbers a, r with $r \neq 1$ that

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r}$$

holds. The series $a + ar + ar^2 + \cdots + ar^n$ is a called a finite ***geometric series***. $\square$

The way I find easiest to use this is to note that if the series $a + ar + ar^2 + \cdots + ar^n$ is continued that the next term would be ar^{n+1}. Therefore if we call the number r the ***ratio*** then

$$a + ar + ar^2 + \cdots + ar^n = \frac{1 - \text{next term}}{1 - \text{ratio}}.$$

Here are some examples

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{1 - \text{next term}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} &= \frac{1 - \text{next term}}{1 - \text{ratio}} \\ &= \frac{1 - (-1/128)}{1 - (-1/2)} = \frac{128 + 1}{128 + 64} = \frac{129}{192}. \end{aligned}$$

For the classical problem[1] of putting one grain rice on the first square of a chess broad, two on the second square, four on the third square, eight on the fourth square: that is doubling the number on each square up until the 64th square, then the total number of grains is

$$1 + 2 + 4 + \cdots + 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Remark 1. The internet tells me that "A single long grain of rice weighs an average of 0.001 ounces (29 mg)." Thus the total weight of the rice on the chess board is $(2^{64}-1)/(1{,}000)$ onces. The number of onces $(2^{64}-1)/(1{,}000)$ in a ton is $2{,}000 \times 16 = 32{,}000$. Therefore the weight in tons of the rice

$$W = (2^{64} - 1)/(1{,}000 \times 32{,}000) = 5.76460752303423 \times 10^{11}.$$

[1]From the Wikipedia article on putting on grains of rice (or wheat) *Wheat and chessboard problem* The wheat and chess problem appears in different stories about the invention of chess. One of them includes the geometric progression problem. The story is first known to have been recorded in 1256 by Ibn Khallikan. Another version has the inventor of chess (in some tellings Sessa, an ancient Indian Minister) request his ruler give him wheat according to the wheat and chessboard problem. The ruler laughs it off as a meager prize for a brilliant invention, only to have court treasurers report the unexpectedly huge number of wheat grains would outstrip the ruler's resources. Versions differ as to whether the inventor becomes a high-ranking advisor or is executed.

The internet also says that the current rate of world rice production is about $P = 7.385477 \times 10^8$ tones/year. At this rate is would take about

$$\frac{W}{P} \approx 780.533$$

years to cover the chess board. $\square$

4.

(a) Find the sum of $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n}$

(b) Find the sum of $P_0(1+r)+P_0(1+r)^2+\cdots P_0(1+r)^n$. (If at the beginning of each year you put P_0 in a bank account that pays interest at a rate of $100r\%$ per year, then this sum is the total after n years. As a check on your answer when $P_0 = 1{,}000$ and $r = .05$, (that is a 5% simple interest) then after 20 years the total is, to the nearest penny, 35,719.25.) $\square$

5. Let

$$f(x) = ax^3 + bx^2 + cx + d$$

be a cubic polynomial. Simplify

$$\frac{f(x) - f(y)}{x - y}$$

by showing that $(x - y)$ can be canceled out of the denominator. What is the result after the cancellation? $\square$

6. Give anther proof of the identity

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$$

by noting

$$x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$$

is a geometric series with first term x^{n-1}, ratio y/x, and the next term would be y^n/x and thus

$$x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = \frac{x^{n-1} - y^n/x}{1 - y/x}$$

which can be simplified to the required identity. $\square$

Summation notation will be used a great deal in this class. We recall the basics about it. The basic notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Thus

$$\sum_{k=0}^{5} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

There is nothing special about using k for the index:

$$\sum_{k=1}^{100} a_k = \sum_{j=1}^{100} a_j = \sum_{\alpha=1}^{100} a_\alpha = \sum_{\smiley=1}^{100} a_{\smiley} = \sum_{\frownie=1}^{100} a_{\frownie}.$$

A basic property of sums we will use is

$$c_1 \sum_{k=m}^{n} a_k + c_2 \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (c_1 a_k + c_2 b_k).$$

We will also want to do changes of index in sum. For example

$$\begin{aligned} \sum_{k=m}^{n} a_k x^{k+3} &= a_m x^{m+3} + a_{m+1} xm + 4 + \cdots + a_{n-1} x^{n+2} + a_n x^{n+3} \\ &= \sum_{m+3}^{n+3} a_{k-3} x^k. \end{aligned}$$

2. The biomomial theorem

We first recall the definition of the ***factorials***. If n is a non-negative integer $n!$ is defined by

$$0! = 1 \quad \text{and for } n \geq 1 \quad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For small values of n we have

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880

n	$n!$
10	3,628,800
11	39,916,800
12	47,9001,600
13	622,7020,800
14	87,178,291,200
15	1,307,674,368,000
16	20,922,789,888,000
17	3556,87,428,096,000
18	6,402,373,705,728,000
19	121,645,100,408,832,000

n	$n!$
20	2,432,902,008,176,640,000
21	51,090,942,171,709,440,000
22	1,124,000,727,777,607,680,000
23	25,852,016,738,884,976,640,000
24	620,448,401,733,239,439,360,000
25	15,511,210,043,330,985,984,000,000
26	403,291,461,126,605,635,584,000,000
27	10,888,869,450,418,352,160,768,000,000
28	304,888,344,611,713,860,501,504,000,000
29	8,841,761,993,739,701,954,543,616,000,000
30	265,252,859,812,191,058,636,308,480,000,000

7. Show that for $n \geq 10$ that $n! \geq 3.6288(10)^{n-4}$. *Hint:* Use that $10! = 3{,}628{,}800 = 3.6288(10)^6$. For example if $n = 15$

$$\begin{aligned} 15! &= 10!(11)(12)(13)(14)(15) \\ &\geq 10!(10)(10)(10)(10)(10) \\ &= 10!(10)^5 \\ &= 3.6288(10)^6(10)^5 \\ &= 3.6288(10)^{11}. \end{aligned}$$

This idea works in general. □

Remark 2. These tables and the last problem make it clear that $n!$ grows very fast. There is a well known approximation, ***Stirling's formula***,

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

which shows that $n!$ grows faster than any exponential function. A more precise form of this was given by Herbert Robbins in 1955:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}.$$

for all positive integers n. Time permitting we will prove some form of this either this term or next term. □

An elementary property of factorials we will use many times is that we get $n!$ by multiplying $(n-1)!$ by n. Thus

$$\begin{aligned} n! &= n((n-1)!) \\ &= n(n-1)((n-2)!) \\ &= n(n-1)(n-2)((n-3)!) \end{aligned}$$

and so on. This especially useful when dealing with fractions involving factorials. For example:

$$\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)n((n-1)!)} = \frac{1}{(n+2)(n+1)n}.$$

Let $n, k \geq 0$ be integers with $0 \leq k \leq n$. Then the ***binomial coefficient*** $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

This is read as "n choose k".

8. Show this this definition implies

$$\binom{n}{k} = \binom{n}{n-k}.$$

$\square$

Also we generally do not have to compute $n!$ to find $\binom{n}{k}$ as lots of terms cancel. For example

$$\binom{100}{3} = \frac{100!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98 \cdot 97!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98}{3!} = 161{,}700.$$

Proposition 3. *The following hold*

$$\begin{aligned}
\binom{n}{0} &= \binom{n}{n} = 1, \\
\binom{n}{1} &= \binom{n}{n-1} = n, \\
\binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2}, \\
\binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

9. Prove this. $\square$

Proposition 4. *The equality*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

holds.

10. Prove this. $\square$

The expression $n(n-1)\cdots(n-k+1)$ comes up often enough that it is worth giving a name. Let $x^{\underline{k}}$ be the ***k-th falling power*** of x. That is

$$x^{\underline{k}} := \begin{cases} 1, & k = 0 \\ x(x-1)\cdots(x-k+1), & k \geq 1. \end{cases}$$

Thus

$$\begin{aligned}
x^{\underline{0}} &= 1 \\
x^{\underline{1}} &= x \\
x^{\underline{2}} &= x(x-1) \\
x^{\underline{3}} &= x(x-1)(x-2) \\
&\vdots \\
x^{\underline{k}} &= \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k \text{ factors}}
\end{aligned}$$

Here is anther basic property of the binomial coefficients.

Proposition 5 (Pascal Identity)**.** *For* $1 \leq k \leq n$ *with* k, n *integers the equality*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

11. Prove this. *Hint:* Here is a special case

$$\begin{aligned}
\binom{12}{7} + \binom{12}{8} &= \frac{12!}{7!\,5!} + \frac{12!}{8!\,4!} \\
&= \frac{12!}{7!\,4!}\left(\frac{1}{5} + \frac{1}{8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{8+5}{5\cdot 8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{13}{5\cdot 8}\right) \\
&= \frac{13!}{8!\,5!} \\
&= \binom{13}{8}
\end{aligned}$$

where we have used $13! = 12! \cdot 13$, $8! = 7! \cdot 8$, and $5! = 4! \cdot 5$. □

If we put the binomial coefficients in a triangular table (***Pascal's triangle***):

$$\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ tells us that any entry is the sum of the two entries directly above. This can be used to compute $\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given by:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The following problem is both interesting in its own right and is a chance to review induction.

12. Let k, n be nonnegative integers with $0 \le k \le n$. Prove the binomial coefficient $\binom{n}{k}$ is an integer. *Hint:* We use induction on n. We can either use $n = 0$ or $n = 1$ as the base case as

$$\binom{0}{0} = 1, \text{ and } \binom{1}{0} = \binom{1}{1} = 1$$

and 1 is an integer. We now use induction on n. Assume that

$$\binom{n}{k} \text{is an integer for } 0 \le k \le n. \tag{1}$$

To complete the induction step we need to show

$$\binom{n+1}{k} \text{is an integer for } 0 \le k \le n+1. \tag{2}$$

This is true for the values $k = 0$ and $k = n+1$ as

$$\binom{n+1}{0} = \binom{n+1}{n+1} = 1.$$

Thus we can assume $1 \le k \le n$. By the Pascal Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

and now use (1) to show that (2) holds to complete the induction. $\square$

One reason the binomial coefficients are important is

Theorem 6 (Binormal Theorem)**.** *For any positive integer n and $x, y \in \mathbb{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We will prove this shortly. So for $n = 5$ we have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Let $x = y = 1$ in this to get

$$\begin{aligned} 2^5 &= (1+1)^5 \\ &= (1)^5 + 5(1)^4(1) + 10(1)^3(1)^2 + 10(1)^2(1)^3 + 5(1)(1)^4 + (1)^5 \\ &= 1 + 5 + 10 + 10 + 5 + 1, \end{aligned}$$

which may not be that interesting of a fact, but the argument lets us see a pattern for something that is interesting

13. Use this idea to show the sum of the numbers $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is 2^n. That is for all positive integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

$\square$

14. Prove for any positive integer n that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: $(1 - 1) = 0$. $\square$

Here is a bit of practice in using the binomial theorem.

15. Expand the following:

(a) $(1 + 2x^3)^4$,
(b) $(x^2 - y^5)^3$.

16. Use induction and the Pascal Identity to prove the Binomial Theorem. *Hint:* Use for the base case that $(x+y)^1 = x+y$. Here is what the induction step from $n = 4$ to $n =$ looks like. Assume that we know that

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}y^4 \\ &= x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\end{aligned}$$

where we have used that $\binom{4}{0} = \binom{4}{4} = 1$. We now want to show the theorem holds for $n = 5$.

$$
\begin{aligned}
(x + y^5) &= (x+y)(x+y)^4 \\
&= (x+y)\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&\qquad + y\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + xy^4 \\
&\qquad + x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}x^1y^4 + y^5 \\
&= x^5 + \left(\binom{4}{0} + \binom{4}{1}\right)x^4y + \left(\binom{4}{1} + \binom{4}{2}\right)x^3y^2 \\
&\qquad + \left(\binom{4}{2} + \binom{4}{3}\right)x^2y^3 + \left(\binom{4}{3} + \binom{4}{4}\right)xy^4 + y^5 \\
&= x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5
\end{aligned}
$$

If you don't like this long hand what of doing it, here is what the same calculation looks like using summation notation. Assume that

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}.$$

Then

$$
\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4\\
&= x(x+y)^4 + y(x+y)^4\\
&= x\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} + y\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k}\\
&= \sum_{k=0}^{4}\binom{4}{k}x^{k+1} y^{4-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k}\\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{4-(k-1)} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k}\\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{5-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k}\\
&= \binom{4}{4}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k}\\
&= \binom{5}{5}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k}\\
&= \binom{5}{5}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\binom{5}{k}x^k y^{5-k}\\
&= \sum_{k=0}^{5}\binom{5}{k}x^k y^{5-k}.
\end{aligned}
$$

where we have done the change of variable $k \mapsto k-1$ in the first sum on line 5, used the Pascal Identity to get to the second to the last line, and used that $\binom{4}{0} = \binom{5}{0} = 1$ and $\binom{4}{4} = \binom{5}{5} = 1$.

Either of these two calculations shows that if the Binomial Theorem holds for $n = 4$ then it holds for $n = 5$. Use a similar calculation to show that if the theorem holds for n, then in holds for $n+1$. $\square$

3. Aside: Elements of the Theory of Finite Differences.

Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function from the integers, $\mathbb{Z}$, to the real numbers, $\mathbb{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

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and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

For example we will be able to show

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

3.1. **The difference operator and the Fundamental Theorem of Summation Theory.**

Definition 7. Let $f\colon \mathbb{Z} \to \mathbb{R}$. Then the ***diffenence***, Δf, of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1) + 2) - (3x + 2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax + b$	a
ar^x	$a(r-1)r^x$

17. Verify these. □

Theorem 8 (**Fundamental Theorem of Summation Theory**)**.** *Let $f\colon \mathbb{Z} \to \mathbb{R}$ and let F be an* ***anti-difference*** *of f. That is $\Delta F = f$. Then for $a, b \in \mathbb{Z}$ with $a < b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}
\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b}(F(k+1) - F(k)) \\
&= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k) \\
&= \big(F(a+1) + F(a+2) + \cdots + F(b) + F(b+1)\big) \\
&\qquad - \big(F(a) + F(a+1) + \cdots + F(b-1) + F(b)\big) \\
&= F(b+1) - F(a)
\end{aligned}$$

as required. □

Theorem 8 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a, r and b are constants).

$f(x)$	$F(x)$
$ax+b$	$a\dfrac{x(x-1)}{2} + bx$
ar^x	$\dfrac{ar^x}{r-1}$

18. Verify these. (You just need to check $F(x+1) - F(x) = f(x)$). □

19. Use that $\dfrac{ar^x}{r-1}$ is the anti-difference of ar^x and Theorem 8 give anther proof of

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

□

3.2. **Falling factorial powers and sums of powers.** For Theorem 8 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 9. For natural number p define the ***falling factorial power*** of $x \in \mathbb{R}$ as $x^{\underline{0}} = 1$ and for $p \geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has p terms.) □

For small values of p this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1\\
x^{\underline{1}} &= x\\
x^{\underline{2}} &= x(x-1)\\
x^{\underline{3}} &= x(x-1)(x-2)\\
x^{\underline{4}} &= x(x-1)(x-2)(x-3)\\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

Proposition 10. *If $f(x) = x^{\underline{p}}$ where p is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

20. Prove this. *Hint:* Here is x(x-1)(x-2)(x-3) that the calculation looks like when $p = 5$.

$$\begin{aligned}
\Delta x^{\underline{5}} &= (x+1)^{\underline{5}} - x^{\underline{5}}\\
&= (x+1)x(x-1)(x-2)(x-3) - x(x-1)(x-2)(x-3)(x-4)\\
&= \big((x+1)-(x-4)\big)x(x-1)(x-2)(x-3)\\
&= 5x^{\underline{4}}.
\end{aligned}$$

$\square$

Remark 11. The formula should remind you of the formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. $\square$

Proposition 12. *If $f(x) = x^{\underline{p}}$ where p is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of f.*

21. Prove this as a corollary of Proposition 10 by noting (by replacing p by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. $\square$

22. Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) $\square$

Proposition 13. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Remark 14. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

23. Prove this. Hint: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 8

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 3.2 to see that $F(1) = 0$. $\square$

Proposition 15. *The equalities*

$$\begin{aligned} x &= x^{\underline{1}} \\ x^2 &= x^{\underline{2}} + x^{\underline{1}} \\ x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\ x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\ x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}} \end{aligned}$$

hold.

24. Verify the first three of these. $\square$

25. Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

HINT: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 13

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\ &= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}. \end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^{n} k^3$. $\square$

3.3. **A couple of trigonometric sums.** For your convenience we recall some trig identities:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ \sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \end{aligned}$$

26. Let θ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show

$$\sin\left(\theta\left(x + \frac{1}{2}\right)\right) - \sin\left(\theta\left(x - \frac{1}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos(\theta x)$$

and therefore

$$F(x) = \frac{\sin\left(\theta\left(x - \frac{1}{2}\right)\right)}{2\sin(\frac{\theta}{2})}.$$

is an anti-difference of

$$f(x) = \cos(\theta x).$$

□

Proposition 16. *If* $\sin\left(\frac{\theta}{2}\right) \neq 0$, *then*

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

27. Use Problem 3.3 and Theorem 8 to prove this. □

There is a similar formula for sums for the sine function.

Proposition 17. *If* $\sin(\frac{\theta}{2}) \neq 0$, *then*

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

28. Prove this. □