NOTES ON ANALYSIS

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1. The real numbers.

1.1. Fields.

A field is an algebraic object where we can do the usual operations of high school algebra. That is addition, subtraction, multiplication, and division.

Definition 1. A ***field*** is a set F with operations[1] $+$, called ***addition***, and $\cdot$, called ***multiplication***, such that

(a) both operations are ***associative***:

$$(x+y)+z = x+(y+z) \qquad x\cdot(y\cdot z) = (x\cdot y)\cdot z$$

for all $x, y, z \in F$.

(b) both operations are ***commutative***:

$$x+y = y+x \qquad x\cdot y = y\cdot x$$

for all $x, y \in F$.

(c) Multiplication ***distributes over addition***:

$$x\cdot(y+z) = x\cdot y + x\cdot z.$$

for all $x, y, z \in F$.

(d) There are additive and multiplicative ***identities***. That is there are $0 \in F$ and $1 \in F$ such that

$$x+0 = x \qquad x\cdot 1 = x$$

for all $x \in F$.

(e) Every element has an ***additive inverse***. That is for every $x \in F$ there is an element $y \in F$ such that

$$x+y = 0.$$

(f) Every nonzero element has a ***multiplicative inverse***. There is for $x \in F$, with $x \neq 0$, there is an element $z \in F$ such that

$$xz = 1.$$

(g) F has at least two elements.

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[1]To be a bit more precise we should call these ***binary operations*** in that that take an ordered pair of elements of F, say (x, y), and each gives a unique output $x + y$ or $x \cdot y$.

This definition requires a bit of comment. First as to the additive identity the definition as it stands does not rule out the possibility that there are two additive identities. that is there are $0, 0' \in F$ with

$$x + 0 = x \quad \text{and} \quad x + 0' = x$$

for all $x \in F$. In this case

$$\begin{aligned} 0' &= 0' + 0 && (x + 0 = x \text{ with } x = 0') \\ &= 0 + 0' && (\text{addition is commutative}) \\ &= 0 && (x + 0' = x \text{ with } x = 0). \end{aligned}$$

So 0 and $0'$ are the same element.

Problem 1. Use a variant on this argument to show that if $1, 1' \in F$ satisfy

$$x \cdot 1 = x \quad \text{and} \quad x \cdot 1' = x$$

for all $x \in F$ that $1 = 1'$. Thus the multiplicative identity is unique. □

We also have that additive inverses are unique. Let $x \in F$ and assume that $y, y' \in F$ such that

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

Then

$$\begin{aligned} y' &= y' + 0 && (x + 0 = x \text{ with } x = y') \\ &= y' + (x + y) && (x + y = 0) \\ &= (y' + x) + y && (\text{associativity of addition}) \\ &= (x + y') + y && (\text{commutative of addition}) \\ &= 0 + y && (x + y' = 0) \\ &= y + 0 && (\text{commutative of addition}) \\ &= y && (x + 0 = x \text{ with } x = y) \end{aligned}$$

Thus the additive inverse of any element any element is unique. From now on we denote the additive inverse of $x \in F$ as $-x$ and use the abbreviation

$$x - y := x + (-y).$$

Proposition 2. *For any $x \in F$ the equality*

$$-(-x) = x$$

holds.

Proof. By definition $-(-x)$ is the additive inverse of $-x$. But we also have

$$\begin{aligned} -x + x &= x + (-x) && (\text{commutative of additive}) \\ &= 0 && (-x \text{ is additive inverse of } x) \end{aligned}$$

This shows that x is also an additive inverse of $-x$. As additive inverses are unique we have $-(-x) = x$. □

Problem 2. Modify the argument above to show that the multiplicative inverse of $x \in F$ with $x \neq 0$ is unique. □

If $x \in F$ and $x \neq 0$ we now denote the unique multiplicative inverse of x by either of the two notations.

$$\text{multiplicative inverse of } x = \frac{1}{x} = x^{-1}.$$

and write

$$yx^{-1} := \frac{y}{x}.$$

Problem 3. Modify one of the arguments above to show if $x \in F$ with $x \neq 0$ then

$$(x^{-1})^{-1} = x.$$

That is

$$\frac{1}{\left(\frac{1}{x}\right)} = x.$$

□

Here are several results that we are so use to seeing that it seems irritating to have to prove them.

Proposition 3. *In a field* $-0 = 0$.

Problem 4. Prove this. *Hint:* $0+0 = 0$ so 0 is the additive inverse of 0. □

Problem 5. In a field, F,

$$x \cdot 0 = 0$$

for all $x \in F$.

Problem 6. Prove this. *Hint:* First show $x \cdot 0 = x \cdot 0 + x \cdot 0$ by justifying the steps in the following.

$$\begin{aligned} x \cdot 0 &= x \cdot (0 + 0) \\ &= x \cdot 0 + x \cdot 0. \end{aligned}$$

Now add the additive inverse of $x \cdot 0$ to both sides of $x \cdot 0 = x \cdot 0 + x \cdot 0$. □

The associativity law implies that for any three elements $x_1, x_2, x_3 \in F$ that

$$(x_1x_2)x_3 = x_1(x_2x_3).$$

As this is the only two ways to group the product of three elements we can write the product of three elements as

$$x_1x_2x_3$$

without ambiguity. There are five ways to group four elements in a product

$$x_1(x_2(x_3x_4)),\ x_1((x_2x_3)x_4),\ (x_1x_2)(x_3x_4),\ (x_1(x_2x_2))x_4,\ ((x_1x_2)x_3)x_4$$

These are all equivalent. We see this by showing they are all the same as $x_1(x_2(x_3x_4))$.

$$
\begin{aligned}
x_1((x_2x_3)x_4) &= x_1(x_2(x_3x_4)) &&\text{as} && (x_2x_3)x_4 = x_2(x_3x_4) \\
(x_1x_2)(x_3x_4) &= x_1(x_2(x_3x_4)) &&\text{as} && (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \\
(x_1(x_2x_3))x_4 &= x_1((x_2x_3)x_4) &&\text{as} && (x_1y)x_4 = x_1(yx_4) \text{ with } y = x_2x_3 \\
&= x_1(x_2(x_3x_4)) &&\text{as} && (x_2x_3)x_4 = x_2(x_3x_4) \\
((x_1x_2)x_3)x_4 &= (x_1x_2)(x_3x_4) &&\text{as} && (yx_3)x_4 = y(x_3x_4) \text{ with } y = x_1x_2 \\
&= x_1(x_2(x_3x_4)) &&\text{as} && (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4
\end{aligned}
$$

So again we can write the product

$$x_1x_2x_3x_4$$

without ambiguity as all the groupings are equal. In light of this the following will most likely not surprise you.

Proposition 4. *Let $x_1x_2, \ldots, x_n$ be elements of the field. Then the associativity law implies that any two groupings of the product $x_1x_2\cdots x_n$ are equal.*

Problem 7. Prove this. *Hint:* Use induction to show that any grouping is equal to the grouping

$$x_1(x_2(x_3(x_4\cdots x_n)\cdots)).$$

This is the grouping where the parenthesis are moved as far to the right as possible. For the rest of this problem call this the ***standard form*** of the product.

Here is the induction step in going from $n = 5$ to $n = 6$. For $n = 5$ the standard form is

$$x_1(x_2(x_3(x_4x_5))).$$

Let p be some grouping of $x_1, x_2, \ldots, x_6$. We first consider the case that p is of the form

$$p = x_1(p_2)$$

where p_2 is a product of $x_2, \ldots, x_6$. Then p_2 is a product of $n = 5$ elements and thus by the induction hypothesis $p_2 = x_2(x_3(x_4(x_5x_6)))$. But then $p = x_1(p_2) = x_1(x_2(x_3(x_4(x_5x_6))))$ can be put in standard form.

This leaves the case where $p = (p_1)(p_2)$ where for some k with $2 \leq k \leq 5$ we have

$$p = (p_1)p_2$$

where p_1 is a product of $x_1, \ldots, x_k$ and p_2 is a product of $x_{k+1}, \ldots, x_6$. Then, as p_1 has less than $n = 6$ factors it can be put in standard. This implies that $p_1 = x_1(q)$ where q is a product of $x_2, \ldots, x_k$. Therefore

$$p = (p_1)p_2 = (x_1q)p_2 = x_1(qp_2).$$

But qp_2 only involves the variables $x_2, \dots, x_6$ so anther application of the induction hypothesis implies that qp_2 can be put standard form. But then $p = x_1(qp_2)$ is in standard form.

To complete the proof you show show that this argument can be used to show that if it is true for n variables, then it is true of $n+1$ variables. □

In light from now on we write products $x_1 x_2 \cdots x_n$ without putting the the parenthesis. There is a similar proposition about parenthesis and and sums, and we will write also write sums as $x_1 + x_2 + \cdots + x_n$ without parenthesis.

The following could be summarized by saying that much of the basic results you know from basic algebra still holds in fields.

Proposition 5. *Let F be a field. Then*

(a) If $a, b, c, d \in F$ and $b, c \neq 0$ then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

(b) If $a, b \in F$ and $a, b = 0$, then $a = 0$ or $b = 0$.

(c) (This is just a useful restatement of part (b).) If $a, b \in F$ and $a, b \neq 0$ then $ab \neq 0$.

(d) If the elements $a_1, a_2, \dots, a_n$ F are all nonzero, then so is the product and

$$(a_1 a_2 \cdots a_{n-1} a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_2^{-1} a_1^{-1}.$$

(e) If $a, b \in F$ and $a^2 = b^2$, then $a = \pm b$.

Problem 8. Prove this. □

Problem 9. Here is anther fact that will likely come up at least one during the term. Let a, b, c, d, e, f be elements of the field F with

$$ad - bc \neq 0.$$

Then the equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

have a unique solution. This solution is

$$x = \frac{ed - bf}{ad - bc}, \qquad y = \frac{af - ec}{ad - bd}.$$

Hint: To find x multiply the first equation by d and the second by b and then subtract the two. A similar trick works to find y. □

2. Square roots.

Theorem 6. *If m is a positive integer that is not a perfect square (that is there is no integer, k, with $k^2 = m$), then $\sqrt{m}$ is irrational.*

Proof. Towards a contradiction assume $x = \sqrt{m}$ is a rational number that is not an integer. Let n be the smallest positive integer such that nx is an integer.

Let $\lfloor x \rfloor$ be the greatest integer in x. Then $0 \le x - \lfloor x \rfloor < 1$. And as x is not an integer $x \ne \lfloor x \rfloor$ and so $0 < x - \lfloor x \rfloor < 1$. Let $p = n(x - \lfloor x \rfloor)$. Then $0 < p < n$ and $p = nx - n\lfloor x \rfloor$ is an integer. But, using that $x^2 = m$,

$$px = n(x - \lfloor x \rfloor)x = nx^2 - (nx)\lfloor x \rfloor) = nm - (nx)\lfloor x \rfloor$$

which is an integer. As $p < n$ this contradicts that n was the smallest positive integer with nx an integer. □

Proposition 7. *Let $a \le x_1, x_2 \le b$. Then*

$$|x_2 - x_1| \le (b - a).$$

Problem 10. Prove this. □

Proposition 8. *Let $A \ge 0$ be such that there is a constant $M > 0$ such that for all $\varepsilon > 0$ the inequality*

$$A \le M\varepsilon$$

holds. Then $A = 0$.

Problem 11. Prove this. □

Theorem 9. *Every positive real number has a unique positive square.*

Problem 12. Prove this. *Hint:* Let a be a positive real number. Let

$$S = \{x \in \mathbb{R} : x \ge 0 \text{ and } x^2 \le a\}$$

(a) Show that S is bounded above. (One way is to note that as $a > 0$ and $x^2 \le a$, then $x^2 \le a < a^2 + 2a + 1 = (a+1)^2$ and therefore $x < a + 1$.)

(b) As S is bounded above the least upper bound axiom applies and thus S has a least upper bound. Let

$$r = \sup(S).$$

Let $0 < \varepsilon < r$. Show
   (i) $(r + \varepsilon)^2 > a$,
   (ii) $(r - \varepsilon)^2 < a$.
   For the first note that $r + \varepsilon > r = \sup(S)$ and so $r + \varepsilon \notin S$. For the second use that $0 < r - \varepsilon < r = \sup(S)$. Thus there is an $x \in S$ with $r - \varepsilon < x$ and therefore $(r - \varepsilon)^2 < x^2$.

(c) Show $(r - \varepsilon)^2 < r^2 < (r + \varepsilon)^2$.

(d) Combine the last two steps to conclude

$$(r - \varepsilon)^2 \le a \le (r + \varepsilon)^2 \text{ and } (r - \varepsilon)^2 < r^2 < (r + \varepsilon)^2.$$

and now use Proposition 7 to conclude

$$|a - r^2| \le (r + \varepsilon)^2 - (r - \varepsilon)^2 = 4r\varepsilon.$$

(e) Now use Proposition 8 to show $r^2 = a$.

(f) Finally show that the positive square root of a is unique. □

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