NOTES ON ANALYSIS

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1. Metric Spaces.

Definition 1. A ***metric space*** is a nonempty set E with a function $d\colon E\times E\to[0,\infty)$ such that for all $p,q,r\in E$ the following hold

(a) $d(p,q)\geq 0$,
(b) $d(p,q)=0$ if and only if $p=q$,
(c) $d(p,q)=d(q,p)$, and
(d) $d(p,r)\leq d(p,q)+d(q,r)$. □

The function d is called the ***distance function*** on E. The condition $d(p,q)=d(q,p)$ is that the distance between points is ***symmetric***. The inequality $d(p,r)\leq d(p,q)+d(q,r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E\subseteq\mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p,q)=|p-q|.$$

Problem 1. Show that this makes E into a metric space. □

We have seen that if $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ are points in $\mathbb{R}^n$ and we define the ***magnitude*** or ***norm*** of p to be

$$\|p\|=\sqrt{p_1^2+\cdots+p_n^2}$$

then the inequality

$$\|p+q\|\leq\|p+q\|$$

holds.

Proposition 2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p,q\in E$ let*

$$d(p,q)=\|p-q\|.$$

Then E with the distance function d is a metric space.

Problem 2. Prove this. □

Here are some inequalities that we will be using later.

Proposition 3. *Let E be a metric space with distance function d and let $x,y,z\in E$. Then*

$$|d(x,y)-d(x,z)|\leq d(y,z).$$

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Problem 3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. □

Proposition 4. *Let E be a metric space with distance function d and $x_1, \ldots, x_n \in E$. Then*

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Problem 4. Prove this. Then draw a picture in the plane showing why this is reasonable. □

Definition 5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

(a) The ***open ball*** of radius r centered at x is

$$B(a, r) := \{x : d(a, x) < r\}.$$

(b) The ***closed ball*** or radius r centered at a is

$$\overline{B}(a, r) := \{x : d(a, x) \leq r\}.$$

□

Definition 6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. □

Somewhat informally this can be restated by saying that S is open if it contains a ball about each of its points.

Proposition 7. *In any metric space E, the sets E and $\varnothing$ are open.* □

Problem 5. Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x, r)$ is an open set.

Problem 6. Prove this. *Hint:* Let $x \in B(a, r)$. Then $d(a, x) < r$. So $\rho := r - d(a, x) > 0$. Show that $B(x, \rho) \subseteq B(a, r)$ □

Proposition 8. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a, b) are open.*

Problem 7. Prove this. □

Proposition 9. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a, r))$, of the closed ball $\overline{B}(a, r)$ is open.*

Proposition 10. *Prove this.* Hint: *If $x \in \mathcal{C}(B(a, r))$, then $d(x, a) > r$. Let $\rho := d(a, x) - r > 0$ and show $B(a, \rho) \subseteq \mathcal{C}(B(a, r))$.* □

Proposition 11. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is any $r > 0$ such $B(x, r) \subseteq U$. But then $B(x, r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x, r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x, r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then

$$B(x,r) \subseteq B(x,r_1) \subseteq U \quad \text{and} \quad B(x,r) \subseteq B(x,r_2) \subseteq V$$

and therefore $B(x, r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. □

Proposition 12. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i\in I} U_i$ is open.

(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 8. Prove this. □

Problem 9. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty}$ is not open.

Definition 13. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. □

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 14. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. □

Proposition 15. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a, r)$ is closed.* □

Problem 10. Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. □

Proposition 16. *If E is a metric space, then every finite subset of E is closed.*

Problem 11. Prove this. □

Problem 12. In the real numbers show that the half open interval $[0, 1)$ is neither open or closed. □

Problem 13. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m, n) = |m - n|$. Note that for this metric space if $m \neq n$ that $d(m, n)$ is a nonzero positive integer and thus $d(m, n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n, r)$ is the one element set $B(n, r) = \{n\}$ and therefore $\{n\}$ is open.

(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n \in S}\{n\}$ and use Proposition 12 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. □

Proposition 17. *Let E be a metric space.*
(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i \in I} F_i$ is closed.
(b) Let $F_1, \ldots, F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 14. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 12. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1) \cap \mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1) \cap \mathcal{C}(F_2))$$

is closed. □

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