Mathematics 554H/701I Homework

Contents

1. Definition of limit in a metric space and some special limits in $\mathbb{R}$.

The topic we have started since the last test is the convergence of sequences.

Definition 1. Let E be a metric space and $\langle p_n\rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots\rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that

$$n > N \quad\implies\quad d(p_n, p) < \varepsilon.$$

In the case we say that the sequence $\langle p_n\rangle_{n=1}^{\infty}$ ***converges*** to p. □

Problem 1. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. □

Problem 2. Write out the proof from the definition that if $\lim_{n\to\infty} x_n = x$ in $\mathbb{R}$, that $\lim_{n\to\infty} -5x_n = -5x$. □

Problem 3. Write out the proof that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in $\mathbb{R}$ that

$$\lim_{n\to\infty}(10x_n - 12y_n) = 10x - 12y.$$

□

We did a proof of the following in class.

Proposition 2. *If $\langle x_n\rangle$ be a convergent sequence in $\mathbb{R}$. Then there is a constant M such that $|x_n| < M$ for all M.* □

Theorem 3. *Let*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$*. Then*

$$\lim_{n\to\infty} x_n y_n = xy.$$

Problem 4. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \qquad \text{(Adding and subtracting trick.)} \\ &\leq |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n\rangle_{n=1}^{\infty}$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n\rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad \text{for all} \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_n > 0$ such that

$$n > N_2 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. □

2. Cauchy sequences, definition of completeness of metric spaces, and proof of the completeness of $\mathbb{R}$.

Proposition 4. *Let* $\langle p_n\rangle_{n=1}^{\infty}$ *be a convergent sequence in the metric space* E*. Let* $\langle p_{n_k}\rangle_{k=1}^{\infty}$ *be a subsequence of this sequence. Then* $\langle p_{n_k}\rangle_{k=1}^{\infty}$ *is also convergent and has the same limit at the original sequence.*

Problem 5. Prove this. *Hint:* For all k we have $n_k \geq k$. □

Definition 5. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of real numbers. Then this sequence is ***monotone increasing*** if and only if $x_n \leq x_{n+1}$ for all n. It is ***monotone decreasing*** if and only if $n_n \geq x_{n_1}$ for all n. It is ***monotone*** if it is either monotone increasing or monotone decreasing. □

Theorem 6. *A bounded monotone sequence in $\mathbb{R}$ is convergent.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a bounded monotone sequence. We first assume that it is monotone increasing. Let

$$S = \{x_n : n = 1, 2, \ldots\}$$

be the set of values of the sequence. As the sequence is bounded, this set is bounded. Therefore, by Least Upper bound Axiom, this set has a least upper bound $b = \sup(S)$. We now show that the sequence converges to b.

Let $\varepsilon > 0$. Then $b - \varepsilon < b$ and b is the least upper bound of S, therefore $b-\varepsilon$ is not an upper bound for S. Therefore there is positive integer N such that $b - \varepsilon < x_N$. Then for any $n > N$ we have

$$\begin{aligned} b - \varepsilon &< x_N \\ &\leq x_n \qquad (x_N \leq x_n \text{ as the sequence is monotone increasing.}) \\ &\leq b \qquad (\text{as } b \text{ is an upper bound for } S \text{ and } x_n \in S.) \end{aligned}$$

Therefore we have $b - \varepsilon < x_n \leq b$ for all $n > N$. Therefore $n > N$ implies $|x_n - b| < \varepsilon$ and thus $\lim_{n\to\infty} x_n = b$. □ □

Problem 6. Modify the last proof so show that if $\langle x_n \rangle_{n=1}^{\infty}$ is bounded and monotone decreasing that it converges to $\inf\{x_n : n = 1, 2, 3, \ldots\}$. □

The following is a very important idea.

Definition 7. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then the sequence is a ***Cauchy sequence*** if and only if for all $\varepsilon > 0$, there is a $N > 0$ such that $m, n > N$ implies $d(p_m, p_n) < \varepsilon$. □

A brief version would be that $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall \varepsilon > 0 \; \exists N > 0 \big[\, m, n > n \quad \Longrightarrow \quad d(p_m, p_n) < \varepsilon \,\big].$$

Proposition 8. *Every convergent sequence is a Cauchy sequence.*

Problem 7. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space and let p be its limit. Let N be so that

$$n > N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

Then show that

$$m, n > n \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

The converse is not true. There are Cauchy sequences that are not convergent.

Problem 8. Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n \rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of E. □

You may feel that the example of the last problem is a bit of a cheat as the sequence does converge in the larger space of all real numbers. And is some sense this is true, given a metric space, E, there is a natural way to expand it to a somewhat larger space that contains the limits of all Cauchy sequences from E.

Proposition 9. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Problem 9. Prove this. *Hint:* Let $\varepsilon > 0$. As the sequence is Cauchy, there is a N such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \frac{\varepsilon}{2}.$$

Let $n > N$, then for any k we have by the triangle inequality that

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p).$$

Now show that it is possible to choose k such that both $d(p_n, p_{n_k})$ and $d(p_{n_k}, p)$ are less than $\varepsilon/2$. $\square$

Theorem 10. *Every sequence of real numbers has a monotone subsequence.*

Problem 10. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Call x_n a ***peak point*** if $x_n \geq x_m$ for all $m > n$. (That is x_n is greater than or equal to all the values that follow it.)

Case 1: *There are infinitely many peak points.* In this case there is an infinite subsequence of the sequence consisting of peak points. Show this subsequence is monotone decreasing.

Case 2: *There are only finitely many peak points.* Let N be the largest n such that x_n is a peak point. Thus if $n > N$ the point x_n is not a peak point and therefore there is $m > n$ with $x_n > x_n$. Let $n_1 = N_1$. Then $n_1 > N$ and so there is a $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. But then $n_2 > N$ and thus there is $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. Continue in this manner to show that there is an infinite increasing subsequence. $\square$

Proposition 11. *Let E be a metric space. Then every Cauchy sequence in E is bounded. (That is the sequence is contained in some ball.)*

Problem 11. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\varepsilon = 1$ (or any other positive number that you like). Then there is $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon = 1.$$

Let $a = N + 1$ and set

$$r = 1 + \max\{1, d(a, x_1), d(a, x_2), \ldots, d(a, x_N)\}.$$

Then show that $p_n \in B(a, r)$ for all n. $\square$

Theorem 12. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Problem 12. Prove this. *Hint:* Let $\langle x_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then by Proposition 11 this sequence is bounded. By Theorem 10 this sequence has a monotone subsequence. By Theorem 6 this monotone subsequence converges. Put these facts together with Proposition 9 to prove that the sequence $\langle x_n\rangle_{n=1}^{\infty}$ converges. □

This property of a metric space, that Cauchy implies convergent, is important enough to give a name.

Definition 13. The metric space E is ***complete*** if and only if every Cauchy sequence in E converges. □

So we can restate Theorem 12 as

Proposition 14. *The real numbers, $\mathbb{R}$, with their usual metric is a complete metric space.* □

3. Completeness of $\mathbb{R}^n$.

We can not get more examples by looking at closed subsets of complete metric spaces.

Proposition 15. *Let E be a metric space and F a closed subset of E. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of points of F that converges in E to some point p. Then $p \in F$. (A nice restatement of this is that a closed set contains all its limit points.)*

Problem 13. Prove this. *Hint:* Towards a contradiction assume that $p \notin F$. Then as F is closed, the compliment $\mathcal{C}(F)$ is open. As $p \in \mathcal{C}F$ by the definition an open set, there is a $r > 0$ such that $B(p,r) \subseteq \mathcal{C}(F)$. But $\lim_{n\to\infty} p_n = p$ and therefore if we let $\varepsilon = r$ there is a $N > 0$ such that $n > 0$ implies $d(p_n, p) < \varepsilon = r$. This this leads to a contradiction. □

Proposition 16. *Let E be a complete metric space and F a closed subset of E. Then F, considered as a metric space in its own right, is complete.*

Problem 14. Prove this. *Hint:* Let $\langle p_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence from F. As E is complete this sequence converges to some point, p, of E. To finish the proof it is enough to show that $p \in F$. □

Recall that we have made $\mathbb{R}^n$ into metric spaces with the metric

$$d(p,q) := \|p - q\|$$

where

$$\|p\| = \|(p_1, p_2, \dots, p_n)\| = \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}.$$

Theorem 17. *With this metric $\mathbb{R}^n$ is complete.*

Problem 15. Prove this in the case of $n = 3$. *Hint:* Here is the proof for $n = 2$. Let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^2$. Note we have the inequality

$$|x_m - x_n| = \sqrt{(x_n - x_n)^2} \leq \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} = d(p_m, p_n)$$

with a similar calculation showing

$$|y_m - y_n| \le d(p_m, p_n).$$

As $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ is Cauchy there is a $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

From the inequalities above this gives

$$m, n > N \quad \text{implies} \quad |x_m - x_n|, |y_m - y_n| \le \|p_m - p_n\| < \varepsilon.$$

Therefore both of the sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ are Cauchy and as $\mathbb{R}$ is complete this implies that they both converge. Thus there are $x, y \in \mathbb{R}^n$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

and thus there are $N_1 > 0$ and $N_2 > 0$ such that

$$\begin{aligned} n > N_1 \quad &\text{implies} \quad |x_n - x| < \frac{\varepsilon}{\sqrt{2}} \\ n > N_2 \quad &\text{implies} \quad |y_n - y| < \frac{\varepsilon}{\sqrt{2}} \end{aligned}$$

Then if $N = \max\{N_1, N_2\}$ and $p = (x, y)$

$$\begin{aligned} n > N \quad \text{implies} \quad \|p_n - p\| &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &< \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} \\ &= \varepsilon. \end{aligned}$$

which shows that the Cauchy sequence $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ has the limit (x, y). As this was an arbitrary Cauchy in $\mathbb{R}^2$, this shows $\mathbb{R}^2$ is complete.

Now you do the proof for $\mathbb{R}^3$. □

The next few definitions, propositions, and problems are practice in using the definitions.

Definition 18. Let E be a metric space and S a subset of E. Then $p \in E$ is an ***adherent point*** of s if and only if every open ball about p contains at least one points of S.

Problem 16*.* To get a feel for what this means, do the following

(a) Show that every point of S is an adherent point of S.
(b) What are the adherent points of the open interval $(0, 1)$?
(c) What are the adherent points of the rational numbers, $\mathbb{Q}$, in the real numbers $\mathbb{R}$? □

Proposition 19. *A set is closed if and only if it contains all its adherent points.*

Problem 17*.* Prove this. *Hint:* Let S be a subset of the metric space E.

(a) First show that if S is closed that it contains all its adherent points. So assume S is closed and p is an adherent point of S. Note for all positive integers n that the open ball $B(p, 1/n)$ contains a point of S as p is an adherent point of S. Let $p_n \in B(p, 1/n) \cap S$. Then show $\lim_{n\to\infty} p_n = p$ and use Proposition 15.

(b) Now show that if S is not closed, then S has an adherent point, p, with $p \notin S$. For if $\mathcal{C}(S)$ is not open and therefore $\mathcal{C}(S)$ has a point $p \in \mathcal{C}(S)$ such that not open ball about p is is contained in $\mathcal{C}(S)$. Show that p is an adherent point of S. □

3.1. **Showing sets are closed.** In what follows we will often want to show that some set is closed. The following gives method for going this that works well in 87.3% of known proofs.

Theorem 20. *Let S be a subset of the metric space E. Then the following are equivalent.*

(a) S is closed.

(b) S contains the limits of its sequences in the sense that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converges, say $x = \lim_{n\to\infty}$, then $x \in S$.

Remark 21. In practice it is the implication (b) $\implies$ (a) that is useful is showing that sets are closed. □

Lemma 22. *Let S be a set in a metric space and p an adherent point of S. Then there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ from S that converges to p.*

Problem 18. Prove this. *Hint:* Let p be an adherent point of S. This means that for every $r > 0$ the ball $B(p, r)$ contains a point of S. For each positive ingeter n let $p_n \in S$ be point of S that is in the ball $B(p, 1/n)$. Now show that $\lim_{n\to\infty} p_n = p$. □

The converse of the last lemma is also true.

Lemma 23. *Let S be a set in a metric space and p a point that is a limit of a sequence of points from S. Then p is an adherent point of S.*

Problem 19. Prove this. *Hint:* Let S be a set in the metric space E and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S that converges to the point $p \in E$. You need to show that p is an adherent point of S, that is if $r > 0$ the ball $B(a, r)$ contains a point of S. As $\lim_{n\to\infty} p_n = p$ we can use $\varepsilon = r$ in the definition of limit to see there is a $N > 0$ such that $n > N$ implies that $d(p_n, p) < r$. □

Problem 20. Prove Theorem 20. *Hint:*

(a) $\implies$ (b). Assume that S is closed and that $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converge to the point p. Use some of the lemmas above to show that p is an adherent point of S and then use that closed sets contain their adherent points. □

(b) $\implies$ (a). Assume that (b) holds. We wish to show that S is closed. It is enough to show that S contains all its adherent points. Let p be an

adherent point of S. Then use one or more of the lemmas above to show that p is a limit of a sequence form S. $\square$

Problem 21. This is an example of using Theorem 20 to to a set is closed. We have seen that if $a, b, c \in \mathbb{R}$ are constants then the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = ax^2 + bx + c$$

satisfies

$$\lim_{n\to\infty} p_n = p \quad \text{implies} \quad \lim_{n\to\infty} f(p_n) = f(p).$$

Let F be a closed subset of $\mathbb{R}$. Show that

$$S := f^{-1}[F] = \{x : f(x) \in F\}$$

is a closed subset of $\mathbb{R}$. *Hint:* Use Theorem 20. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence of points from S with $\lim_{n\to\infty} = p$. All we need to to is show that $p \in S$. By the definition of S we have $f(p_n) \in F$. Also we have $f(p) = \lim_{n\to\infty} f(p_n)$. Now use some of the results above to show that $f(p)$ as an adherent point of F and as F is closed that this implies $f(p) \in F$. $\square$

4. Sequential compactness and the Bolzano-Weierstrass Theorem.

Definition 24. A subset S of a metric space is ***sequentially compact*** if and only if every sequence $\langle p_n\rangle_{n=1}^{\infty}$ of points from S has a subsequence that converges to a point of S. $\square$

Problem 22. Let S be a finite subset of a metric space. Then S is sequentially compact. *Hint:* Let $S = \{s_1 s_2, \ldots, s_m\}$. For each $j \in \{1, 2, \ldots, m\}$ let $\mathcal{N}_j = \{n : p_n = s_j\}$. As the union of the sets $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_m$ it the infinite set $\mathbb{N} = \{1, 2, 3, \ldots\}$ at least one of them is infinite. Say that $\mathcal{N}_j$ is infinite, $\mathcal{N}_j = \{n_1, n_2, n_3, \ldots\}$ and consider the subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$. $\square$

Theorem 25 (BolzanoWeierstrass Theorem)**.** *Every closed bounded subset of* $\mathbb{R}$ *is sequentially compact.*

Problem 23. Prove this. *Hint:* Let S be a closed bounded subset of $\mathbb{R}$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of points from S. Then the sequence is bounded (because S is bounded). Also (Theorem 10) this sequence has a monotone subsequence. At some point in finishing the proof you will need to use Proposition 15. $\square$

Corollary 26 (BolzanoWeierstrass Theorem for sequences)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof. Let $\langle x_n\rangle_{n=1}^{\infty}$ be bounded sequence in $\mathbb{R}$. As the sequence is bounded there is a closed ball $\overline{B}(a, r) = [a - r, a + r]$ that contains $\langle x_n\rangle_{n=1}^{\infty}$. The set $\overline{B}(a, r)$ is a closed bounded subset of $\mathbb{R}$ and so by the Bolzano-Weierstrass the $\langle x_n\rangle_{n=1}^{\infty}\langle x_n\rangle_{n=1}^{\infty}$ has a convergent subsequence. $\square$

Theorem 27 (General BolzanoWeierstrass Theorem). *Every closed bounded subset of $\mathbb{R}^n$ is sequentially compact.*

Lemma 28. *Let $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n) \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$. Then this sequence converges if and only if both the sequences*

$$\langle x_n \rangle_{n=1}^{\infty} \quad \textit{and} \quad \langle y_n \rangle_{n=1}^{\infty}$$

converge.

Proof. Let $p = (x, y) \in \mathbb{R}^2$. Then, as we saw in Problem 15 the inequalities

$$|x_n - x|,\ |y_n - y| \leq \sqrt{(x - x_n)^2 + (y_n - y)^2} = d(p_n, p)$$

Therefore if $\langle p_n \rangle_{n=1}^{\infty}$ converges, as $\lim_{n\to\infty} p_n = p$ then for any $\varepsilon > 0$ there is a $N > 0$ such that $n > N$ implies $d(p_n, p) < \varepsilon$. Therefore for this N we have

$$n > N \quad \text{implies} \quad |x_n - x|, |y_n - y| \leq d(p_n, p) < \varepsilon$$

and therefore we have $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Conversely assume that both the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, say $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Therefore there are N_1 and N_2 such that

$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and}$$

Thus is $N = \max\{N_1, N_2\}$ we have, just as in Problem 15,

$$n > N \quad \text{implies} \quad d(p_n, p) < \varepsilon$$

which shows that $\langle p_n \rangle_{n=1}^{\infty}$ converges. Now show the limit is in S. □

Lemma 29. *Let $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n, z_n) \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^3$. Then this sequence converges if and only if all three of they sequences*

$$\lim_{n\to\infty} x_n, \quad \lim_{n\to\infty} y_n, \quad \textit{and} \quad \lim_{n\to\infty} z_n$$

converge.

Problem 24. Prove Theorem 27 for $n = 3$. *Hint:* Here is the proof for $n = 2$. Let S be a closed bouneded subset of $\mathbb{R}^n$ and $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n \rangle_{n=1}^{\infty}$ a sequence in S. As S is bounded the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is bounded. But

$$|x_n|,\ |y_n| \leq \sqrt{|x_n|^2 + |y_n|^2} = d(\vec{0}, p_n)$$

and therefore both of the sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ are bounded. As the sequence $\langle x_n \rangle_{n=1}^{\infty}$ is bounded by Corollary 26 it has a convergent subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$. The sequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence of the bounded sequence $\langle y_n \rangle_{n=1}^{\infty}$ and therefore $\langle y_{n_k} \rangle_{k=1}^{\infty}$ is also bounded. Therefore we can use Corollary 26 again to get a subsequence $\langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$ of the subsequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ $\langle y_{n_k} \rangle_{k=1}^{\infty}$ such that $\langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$ converges. Now note that the subsequence $\langle x_{n_{k_j}} \rangle_{j=1}^{\infty}$ is a convergent subsequence of the convergent sequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$. But a subsequence of a convergent subsequence is convergent

(Proposition 4) and therefore $lax_{n_{k_j}}\rangle_{j=1}^{\infty}$ is convergent. But then both of the sequences

$$\langle x_{n_{k_j}}\rangle_{j=1}^{\infty} \quad \text{and} \quad \langle y_{n_{k_j}}\rangle_{j=1}^{\infty}$$

converge and therefore by Lemma 28 this implies the sequence

$$\langle p_{n_{k_j}}\rangle_{j=1}^{\infty} = \langle (x_{n_{k_j}}, y_{n_{k_j}})\rangle_{j=1}^{\infty}$$

converges. Let $p = \lim_{n\to\infty} p_{n_{k_j}}$. Then, as S is closed, Proposition 15 implies $p \in S$. As $\langle p_n\rangle_{n=1}^{\infty}$ was any sequence from the closed bounded set, S, this shows that every sequence from a closed bounded subset of $\mathbb{R}^2$ has a subsequence that converges to a point of S. Therefore closed bounded subsets of $\mathbb{R}^n$ are sequentially compact. □

Corollary 30 (General Bolzano-Weierstrass for sequences)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Problem 25*.* Prove this. *Hint:* See the proof of Corollary 26. □

Proposition 31. *Every sequentially compact subset of a metric space is closed and bounded.*

Problem 26*.* Prove this. *Hint:* Let S be sequentially compact in E. First show that S is bounded. Towards a contradiction assume that it is not bounded. Let q be any point of S. Because S is not bounded there for each positive integer n there is a point $p_n \in S$ with $d(q, p_n) > n$. The set S is sequentially compact and therefore the sequence $\langle p_n\rangle_{n=1}^{\infty}$ has a convergent subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$. Let $p = \lim_{k\to\infty} p_{n_k}$. Then using $\varepsilon = 1$ in the definition of limit we have that there is a $K > 0$ such that $k > K$ implies that $d(p, p_{n_k}) < 1$. Whence for all $k > K$ we have by the triangle inequality

$$n_k < d(q, p_{n_k}) \le d(q, p) + d(p, p_{n_k}) < d(q, p) + 1$$

which gives a contradiction (why?)

Now use sequential compactness to show S is closed. One way is to show that sequential compactness implies that S contains all its adherent points. □

Remark 32*.* We have seen, Theorem 27, that every closed bounded of $\mathbb{R}^n$ is sequentially compact. And the last proposition shows that a sequentially compact subset is closed and bounded. But it is important to realize that not all closed bounded subsets of all subsets of all metric spaces are sequentially compact. The next problem give an example. □

Problem 27*.* Let $E = (0, \infty)$ and let $S = (0, 1]$. Here we are using the metric $d(x, y) = |x - y|$. Show that S is a closed bounded subset of E, but that S is not sequentially compact. □

5. Open covers and the Lebesgue covering lemma.

We recall a bit of set theory. Let E a set and $\mathcal{U}$ a collection of subsets of E. (At bit more formally if $U \in \mathcal{U}$ then $U \subseteq E$.) The ***union*** of $\mathcal{U}$ is

$$\bigcup \mathcal{U} = \{x : x \in U \text{ for at least one } U \in \mathcal{U}\}.$$

We will sometimes use the notation

$$\bigcup_{U \in \mathcal{U}} U$$

or some trivial variants of this notation. For example

$$\bigcup_{n=1}^{\infty} (-n, n) = (-\infty, \infty)$$

or

$$\bigcup_{x \in [0,1]} (x-1, x+1) = (-1, 2).$$

Of course there is the ***intersection*** of $\mathcal{U}$ which it

$$\bigcap \mathcal{U} = \{x : x \in U \text{ for all } U \in \mathcal{U}\}.$$

which can also be written as

$$\bigcap_{U \in \mathcal{U}} U$$

with such variants as

$$\bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n) = [a, b]$$

(which gives anther example of an infinite intersection of open sets not being open), and

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \varnothing.$$

Definition 33. Let E be a metric space and $S \subseteq E$. Then $\mathcal{U}$ is an ***open cover*** of S if and only if the following hold

(a) Each element, U, of $\mathcal{U}$ is an open subset of E.
(b) $S \subseteq \bigcup \mathcal{U}$. $\square$

Anther way to say $S \subseteq \cup \mathcal{U}$ is that for all $x \in S$ there is an $U \in \mathcal{U}$ with $x \in U$. This is nothing more than a restatement of the definition of the union, but in practice is how we often work with open covers.

Theorem 34 (Lebesgue Covering Theorem)**.** *Let S be a sequentially compact subset of the metric space E and let $\mathcal{U}$ be an open over of S. Then there is a $r > 0$ (often called a **Lebesgue number** of the cover) such that for all $x \in S$ there are is a $U \in \mathcal{U}$ with $B(x, r) \subseteq U$.*

A restatement is that given an open cover $\mathcal{U}$ of a sequentially compact set S there is a $r > 0$ (which depends on both S and $\mathcal{U}$) such that every point of S is contained in a ball of radius r that is contained in some open set $U \in \mathcal{U}$. In practice this means that in working with open covers, we can sometimes replace them with a cover by balls all with the same radius.

Problem 28. Prove Theorem 34. *Hint:* Towards a contradiction assume that there is an open cover $\mathcal{U}$ of a sequentially compact set S where the Lebesgue Covering Theorem does not hold. This means that for all $r > 0$ there is a point $x \in S$ such that the ball $B(x, r)$ is not contained in any $U \in \mathcal{U}$.

For each positive integer n let $x_n \in S$ be a point where the ball $B(x_n, 1/n)$ is not contained in any of the sets $U \in \mathcal{U}$. As S is sequentially compact, the sequence $\langle x_n \rangle_{n=1}^{\infty}$ has a convergent subsequence, $\langle x_{n_k} \rangle_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = x$ where $x \in S$. As $x \in S$ and $\mathcal{U}$ is an open cover of S there is some $U \in \mathcal{U}$ with $x \in U$. As U is open there is a $r > 0$ such that $B(x, r) \subseteq U$. Because $\lim_{k\to\infty} x_{n_k} = x$ there is a $N > 0$ such that

$$k > N \quad \text{implies} \quad d(x_{n_k}, x) < \frac{r}{2}.$$

Now show that if we choose k such that both $k > N$ and $1/n_k < r/2$ hold then

$$B(x_{n_k}, 1/n_k) \subseteq B(x, r) \subseteq U$$

and explain why this leads to a contradiction. $\square$

6. Open covers and compactness.

Definition 35. Let S be a subset of the metric space E. Then S is ***compact*** if and only if every open cover of S has a finite subcover. Explicitly the means that if $\mathcal{U}$ is an open cover of S then there is a finite set $\{U_1, U_2, \ldots, U_m\} \subseteq \mathcal{U}$ with

$$S \subseteq U_1 \cup U_2 \cup \cdots \cup U_m$$

$\square$

Theorem 36. *Every sequentially compact set in a metric space is compact.*

Problem 29. Prove this. *Hint:* Towards a contradiction assume that S is a sequentially compact subset of some the metric space E that is not compact. That is there is some open cover of $\mathcal{U}$ of S that as no finite subcover. Let r be a Lebesgue number for this open cover. That is for every $p \in S$ there is some $U \in \mathcal{U}$ such that $B(p, r) \subseteq U$. We know that such an r exists by the Lebesgue Covering Theorem 34. Define a sequence of points $p_1, p_2, p_3, \ldots \in S$ and a sequence $U_1, U_2, U_3, \ldots \in \mathcal{U}$ follows. Let p_1 be any element of S. Then there is a $U_1 \in \mathcal{U}$ such that $B(p_1, r) \subseteq U_1$. Now assume that $p_1, 2_2, \ldots, p_n \in S$ and $U_1, U_2, \ldots, U_n \in \mathcal{U}$ have been defined such that

$$p_j \notin U_1 \cup U_2 \cup \cdots \cup U_{j-1} \quad \text{and} \quad B(x_j, r) \subseteq U_j$$

for $j = 1, 2, \ldots, n$. Now

(a) Explain why there is an $p_{n+1} \in S$ such that $p_{n+1} \notin U_1 \cup U_1 \cup \cdots U_n$.
(b) There is a $U_{n+1} \in \mathcal{U}$ such that $B(p_{n+1}, r) \subseteq U_{n+1}$.

Finish the proof by showing that if $m \neq n$, say $m < n$, then $p_n \notin U_m$ and $B(p_m, r) \subseteq U_m$ implies that $d(p_m, p_n) \geq r$ and thus the sequence $\langle p_n \rangle_{n=1}^{\infty}$ has no convergent subsequence (if $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence has $d(p_{x_k}, p_{x_\ell}) \geq r$ for $k \neq \ell$. Use this to show the subsequence is not Cauchy) which contradicts that S is sequentially compact. $\square$

The converse of the last Theorem is also true.

Theorem 37. *Every compact set in a metric space is sequentially compact.*

Lemma 38. *Let S be a compact set in a metric space and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then there is a point $p \in S$ such that for all $r > 0$ the set*

$$\{n : p_n \in B(p, r)\}$$

is infinite.

Problem 30. Prove this. *Hint:* This is a very typical use of compactness in a proof. Towards a contradiction assume that there is a compact set S where this does not hold. Then for each $x \in S$ there is a $r_x > 0$ such that $\{n : p_n \in B(x, r_x)\}$ is finite. Set

$$\mathcal{U} = \{B(x, r_x) : x \in S\}.$$

Show that $\mathcal{U}$ is an open cover of S. By compactness there is a finite subcover, say that

$$S \subseteq B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \cdots \cup B(x_m, r_{x_m}).$$

Now for each natural number $n \in \mathbb{N}$ we have that $p_n \in B(x_j, r_{x_j})$ for at least one $j \in \{1, 2, \ldots, m\}$. Thus, by the pigeon hole principle, there is at least one j where the set $\{n : p_n \in B(x_j, r_{x_j})\}$ is infinite. Explain why this is a contradiction. $\square$

Problem 31. Prove this. *Hint:* Let S be a compact set in a metric space E. We wish to show that every sequence $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to a point of S. Towards this end let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. By Lemma 38 there is a point $p \in S$ such that for all $r > 0$ the set of $n \in \mathbb{N}$ with $p_n \in B(x, r)$ is infinite. Show this implies $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to p. $\square$