

## Mathematics 554H/701I Homework

We now start come the the last big topic we will cover this term, which is continuous maps between metric spaces.

**Definition 1.** Let  $E$  and  $E'$  be metric spaces and  $f: E \rightarrow E'$  a function from  $E$  to  $E'$ . Let  $p_0 \in E$ . Then  $f$  is **continuous** at  $p_0$  if and only if for all  $\varepsilon > 0$  there is a  $\delta > 0$  such that for  $p \in E$

$$d(p, p_0) < \delta \quad \text{implies} \quad d(f(p), f(p_0)) < \varepsilon.$$

□

*Example 2.* Here is an example of showing something is continuous. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function

$$f(x) = 3x + 5$$

Then  $f$  is continuous at every point of  $\mathbb{R}$ . To see this let  $x_0 \in \mathbb{R}$  and let  $\varepsilon > 0$ . Let  $\delta = \varepsilon/3$ . Then if  $|x - x_0| < \delta$  we have

$$\begin{aligned} |f(x) - f(x_0)| &= |3x + 5 - (3x_0 + 5)| \\ &= |3(x - x_0)| \\ &= 3|x - x_0| \\ &< 3\delta \\ &= \varepsilon. \end{aligned}$$

**Proposition 3.** Let  $E$  be a metric space and  $f: E \rightarrow E$  the identity map, that is  $f(p) = p$  for all  $p \in E$ . Then  $f$  is continuous.

**Problem 1.** Prove this. □

**Problem 2.** Let  $E$  be a metric space.

(a) Let  $p, x_0, q \in E$  show that

$$|d(q, x_0) - d(p, x_0)| \leq d(p, q).$$

(b) Let  $x_0 \in E$  and define  $f(p)$  to be the distance of  $p$  from  $x_0$ , that is  $f(p) = d(p, x_0)$ . Show that  $f$  is continuous at all points of  $E$ . *Hint:* Use part (a) to show  $|f(p) - f(q)| \leq d(p, q)$ . □

Recall that on  $\mathbb{R}^n$  we have defined the inner product

$$a \cdot b = \sum_{j=1}^n a_j b_j$$

where  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$ . This was used to define the norm on  $\mathbb{R}^n$  as

$$\|a\| = \sqrt{a \cdot a}.$$

This in turn was used to define the distance function on  $\mathbb{R}^n$  by

$$d(p, q) = \|p - q\|.$$

Also recall that we have the Cauchy-Schwartz inequality:

$$|a \cdot b| \leq \|a\| \|b\|.$$

**Problem 3.** Let  $a \in \mathbb{R}^n$  and define the function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$f(x) = a \cdot x.$$

Show that  $f$  is continuous at all points of  $\mathbb{R}^n$ . *Hint:* Let  $x, x_0 \in \mathbb{R}^n$  then

$$f(x) - f(x_0) = a \cdot x - a \cdot x_0 = a(x - x_0).$$

Now use the Cauchy-Schwartz inequality to show  $|f(x) - f(x_0)| \leq \|a\| \|x - x_0\|$ .  $\square$

**Problem 4.** Define the functions  $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $f(x, y) = x$  and  $g(x, y) = y$ . Show that  $f$  and  $g$  are continuous. *Hint:* As the two proofs are the same, it is enough to show that  $f$  is continuous. Let  $a = (1, 0)$ , the  $f(x, y) = (x, y) \cdot a$  so one way to do this is to reduce it to the previous problem.  $\square$

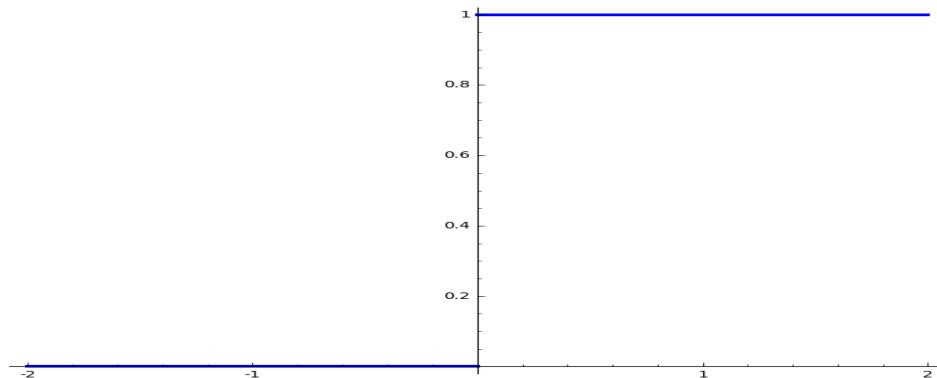
We now give examples of some functions that are not continuous. We first record what it means for a function to not be continuous at a point.

**Negation of Definition of Continuity.** 4. Let  $f: E \rightarrow E'$  be a map between metric spaces. Let  $p_0 \in E$ . Then  $f$  is **discontinuous** at  $p_0$  if and only if there is a  $\varepsilon > 0$  such that for all  $\delta > 0$  there is a  $p \in E$  with  $d(p, p_0) < \delta$  and  $d'(f(p), f(p_0)) \geq \varepsilon$ .  $\square$

We now look at the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & 0 < x. \end{cases}$$

which has the graph:



We now show this is discontinuous at  $x = 0$ . Let  $\varepsilon = 1/2$ . Then for any  $\delta > 0$  there is an  $x > 0$  with  $0 < x < \delta$ . Then  $x > 0$  and so  $f(x) = 1$ . As  $f(0) = 0$  we have  $|f(x) - f(0)| = |1 - 0| = 1 > \varepsilon$  as required.

Here is a more exotic example.

**Problem 5.** Define a function by

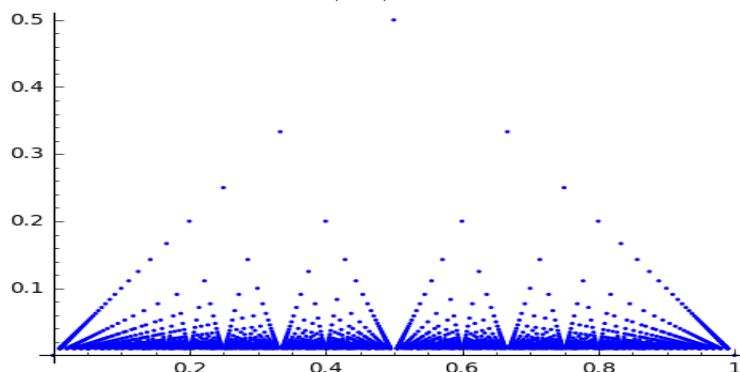
$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}; \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

That is  $f(x)$  is one with  $x$  is a rational number, and  $f(x)$  is zero when  $x$  is irrational. Show that  $f$  is discontinuous at all points of  $\mathbb{R}$ .

**Problem 6** (Optional). Define a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ is rational in lowest terms;} \\ 0, & x \text{ is irrational.} \end{cases}$$

Here is the graph for rationals in  $(0, 1)$  with denominators less than 100.



Show that  $f$  is continuous at all irrational points and discontinuous at all rational points.  $\square$

**Problem 7.** Let  $f: (0, \infty) \rightarrow \mathbb{R}$  be defined by

$$f(x) = \sqrt{x}$$

then show  $f$  is continuous at  $x = 1$ .

*Solution:* We first note that

$$|f(x) - f(1)| = |\sqrt{x} - 1| = \left| \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(\sqrt{x} + 1)} \right| = \left| \frac{x - 1}{\sqrt{x} + 1} \right| \leq \left| \frac{x - 1}{0 + 1} \right| = |x - 1|.$$

Now let  $\varepsilon > 0$  and let  $\delta = \varepsilon$ . Then if  $|x - 1| < \delta$  implies

$$|f(x) - f(1)| \leq |x - 1| < \delta = \varepsilon$$

which is just what is needed to show that  $f(x)$  is continuous at  $x = 1$ .  $\square$

**Problem 8.** Let  $f: (0, \infty) \rightarrow \mathbb{R}$  be defined by

$$f(x) = \sqrt{x}.$$

Show  $f$  is continuous at  $x = a$  for any  $a > 0$ .  $\square$

**Theorem 5.** Let  $E$  be a metric space and  $f, g: E \rightarrow \mathbb{R}$  be functions and  $c_1, c_2 \in \mathbb{R}$  constants. Assume  $f$  and  $g$  are continuous at  $p_0$ . Then the functions  $c_1f + c_2g$  and  $fg$  are continuous at  $p_0$ . If  $g(p_0) \neq 0$  then the quotient  $\frac{f}{g}$  is continuous at  $p_0$ .  $\square$

*Proof.* See the class notes or the text.  $\square$   $\square$

**Proposition 6.** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

*Then  $f$  is continuous at all points of  $\mathbb{R}$ .*

**Problem 9.** Prove this. *Hint:* Probably the easiest way is by induction on  $n$ . The base of the induction is  $n = 0$  in which case  $f(x) = a_0$  is a constant which is clearly continuous. Or we can use the base case of  $n = 1$  in which case  $f(x) = a_1 x + a_0$  is linear and we have done this in class.

Here is what the induction step from  $n = 4$  to  $n = 5$  looks like. Assume that we know that all polynomials of degree 4 are continuous and let

$$f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

be a polynomial of degree 5. Write it as

$$\begin{aligned} f(x) &= x(a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1) + a_0 \\ &= xg(x) + a_0 \end{aligned}$$

where  $g(x) = a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1$  is a polynomial of degree 4. By the induction hypothesis  $g(x)$  is continuous and the function  $x$  is continuous. Therefore  $f$  is of the form

$$f = (\text{continuous function}) \times (\text{continuous function}) + (\text{constant})$$

and therefore  $f$  is continuous. Use this idea to do the general induction step.  $\square$