Mathematics 554H/701I Homework

We now come to the last big theorems of the term. We first recall the following fact about continuous functions that has been useful for proving results about connected sets (and will shortly be useful in proving results about continuous functions).

Theorem 1. *Let $f\colon E \to E'$ be a map between metric spaces. Then f is continuous if and only if for all open sets $U \subseteq E'$ the preimage $f^{-1}[U]$ is open in E.* $\square$

Problem 1*.* Note that this does *not* say that the continuous image of an open set is open. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the the continuous function $f(x) = x^2$. Show that the image $f[(-1,1)]$ is not open even though $(-1,1)$ is open. $\square$

Recall that $\mathcal{U}$ is an ***open cover*** of the metric space E if each $U \in \mathcal{U}$ is an open subset of E and each $p \in E$ is an element of at least one $U \in \mathcal{U}$.

Proposition 2. *Let $f\colon E \to E'$ be a continuous map between metric spaces. Let $\mathcal{U}'$ be an open cover of the image $f[E]$. Then*

$$\mathcal{U} = \{f^{-1}[U] : U \in \mathcal{U}'\}$$

is an open cover of E.

Problem 2*.* Prove this. $\square$

Theorem 3 (Continuous Images of Compact Sets are Compact)**.** *Let E be a compact metric space and $f\colon E \to E'$ a continuous map between metric spaces. Then the image $f[E]$ is compact.*

Problem 3*.* Prove this. *Hint:* Let $\mathcal{U}'$ be an open cover of $f[E]$. We need to show that $\mathcal{U}'$ has a finite subcover. As in the last proposition let $\mathcal{U} = \{f^{-1}[U] : U \in \mathcal{U}'\}$. Then this is an open cover of E. But E is compact so there is a finite set $\mathcal{U}_0 = \{f^{-1}[U_1], f^{-1}[U_2], \ldots, f^{-1}[U_n]\} \subseteq \mathcal{U}$ such that

$$E = f^{-1}[U_1] \cup f^{-1}[U_2] \cup \cdots \cup f^{-1}[U_n].$$

Now show that $\{U_1, U_2, \ldots, U_n\}$ is a finite subset of $\mathcal{U}'$ that covers $f[E]$. This shows that every open cover, $\mathcal{U}'$, of $f[E]$ has a finite subcover and thus $f[E]$ is compact. $\square$

We recall the following from earlier in the term.

Theorem 4. *A subset S is $\mathbb{R}$ is compact if and only if it is closed and bounded.* $\square$

We have also seen that the following true:

Proposition 5. *Let $S \subseteq \mathbb{R}$ be a compact set. Then S has a largest and smallest element.*

Proof. Because S is compact, it is closed and bounded. As S is bounded, it has a least upper bound. Let $\beta = \sup(S)$. As S is closed it contains β. We recall the proof of this. Towards a contradiction assume that $\beta \notin S$. Then the compliment $\mathcal{C}(S)$ is open and so for some $r > 0$ we have $B(\beta, r) = (\beta - r, \beta + r) \subseteq \mathcal{C}(S)$. But this implies that $\beta - r < \beta$ is an upper bound for S (draw picture) contracting that β is the least upper bound for S. Thus $\beta \in S$. And β is the largest element of S as for all $x \in S$ we have $x \leq \beta$.

A similar proof shows that $\alpha = \inf(S) \in S$ and thus α is the smallest element of S. □

The important part of the last proposition is that $\sup(S)$ and $\inf(S)$ are elements of S when S is a compact subset of $\mathbb{R}$.

Theorem 6 (Continuous Functions on Compact Sets Achieve Their Maximum and Minimum)**.** *Let E be a compact metric space and $f\colon E \to \mathbb{R}$. Then f achieves its maximum and minimum. The is there are points $x_0, x_1 \in E$ such that*

$$f(x_0) \leq f(x) \leq f(x_1).$$

(Thus $f(x_0)$ is the minimum value of f and $f(x_1)$ is the maximum value. The element x_0 is a ***minimizer*** *of f and x_1 is a* ***maximizer****.)*

Problem 4. Prove this. *Hint:* As E is compact the image $S := f[E]$ is compact by Theorem 3. By Proposition 5 we have that $f[E]$ has a largest element, β, and a smallest element, α. As $\alpha, \beta \in f[E]$ there are points $x_0, x_1 \in E$ with $f(x_0) = \alpha$ and $f(x_1) = \beta$. Show that x_0 and x_1 are are required. □

Finally we wish to relate the continuity of functions with limits of sequences.

Theorem 7. *Let $f\colon E \to E'$ be a map between metric and $f\colon E \to E'$ a function that is continuous at the point p_0. Then for any $\langle p_n \rangle_{n=1}^{\infty}$ vsequence in E with $\lim_{n\to\infty} p_n = p_n$ we have*

$$\lim_{n\to\infty} f(p_n) = f(p_0).$$

Problem 5. Prove this. *Hint:* Let $\varepsilon > 0$. Then because f is continuous at p_0 there is a $\delta > 0$ such that for any $p \in E$ with $d(p, p_0) < \delta$ we have $d'(f(p), f(p_0)) < \varepsilon$. Because $\lim_{n\to\infty} p_n = p_0$ there is a $N > 0$ such that $n > N$ implies $d(p_n, p_0) < \delta$. Now show that $n > N$ implies that $d'(f(p_n), f(p_0)) < \varepsilon$ which is exactly what is needed to show that $\lim_{n\to\infty} f(p_n) = f(p_0)$. □

The converse of the last theorem is also ture.

Theorem 8. *Let $f\colon E \to E'$ be a map between metric spaces. Let $p_0 \in E$ and assume that for every sequence $\langle p_n \rangle_{n=1}^{\infty}$ in E with $\lim_{n\to\infty} p_n = p_0$ that*

$$\lim_{n\to\infty} f(p_n) = f(p_0).$$

Then f is continuous at p_0.

Problem 6. Prove this. *Hint:* Towards a contraction assume that the limit condition holds, but that f is not continuous at p_0. Then there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0) < \delta$ but $d'(f(p), f(p_0)) \geq \varepsilon$. Letting $\delta = 1/n$ we get a point $p_n \in E$ with $d(p_n, p_0) < 1/n$ and $d'(f(p_n), f(p_)) \geq \varepsilon$. Now show that $\lim_{n\to\infty} p_n = p_0$, but that $\lim_{n\to\infty} f(p_n) \neq f(p_0)$, which is the desired contradiction. □

We can combine these results to get:

Theorem 9. *Let $f\colon E \to E'$ be a map between metric space. Then f is continuous if and only if f preserves the limits of sequences. Explicitly this f is continuous if and only if whenever* $\lim_{n\to\infty} p_n = p_0$ *for a sequence in E that also* $\lim_{n\to\infty} f(p_n) = f(p_0)$ *in E'.*

Problem 7. Prove this. □

We can use these results to given anther proof that the continuous image of a compact set is compact. Recall that the metric space E is ***sequentially*** compact if and only if every sequence $\langle p_n\rangle_{n=1}^{\infty}$ in E and a convergent subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$.

Theorem 10. *Let $f\colon E \to E'$ be a continuous map between metric spaces. Assume that E is sequentially compact. Then the image $f[E]$ is sequentially compact.*

Problem 8. Prove this. *Hint:* Let $\langle q_n\rangle_{n=1}^{\infty}$ is a sequence in $f[E]$. We wish to show that this has a convergent subsequence. As $q_n \in f[E]$ we have that is a point $p_n \in E$ with $f(p_n) = q_n$. As E is sequentially compact the sequence $\langle p_n\rangle_{n=1}^{\infty}$ has some convergent subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$, say $\lim_{k\to\infty} p_{n_k} = p_0$. Now use Theorem 7 to show $\lim_{k\to\infty} f(p_{n_k}) = f(p_0)$. Therefore $\langle q_n\rangle_{n=1}^{\infty} = \langle f(p_n)\rangle_{n=1}^{\infty}$ has a convergent subsequence as required. □

One of the hardest results we have shown in this class is

Theorem 11. *A metric space, E, is sequentially compact if and only if it is compact.* □

In light of this we have that Theorems 3 and 10 really say the same thing. So we have two proofs of the result that the continuous image of a compact set is compact.