Mathematics 554H/703I Test 2 Name:**Answer Key.**

1. (a) Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence of points in a metric space. Define $\lim_{n\to\infty} p_n = p$.

Solution: For all $\varepsilon > 0$ there is a positive integer N such that $n > N$ implies $d(p_n, p) < \varepsilon$. □

(b) Let $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n$. Prove directly from the definition of limit that

$$\lim_{n\to\infty}(2x_n + 3y_n) = 2x + 3y.$$

Solution: Let $\varepsilon > 0$. By the definition of $\lim_{n\to\infty} x_n = x$ there is a $N_1 > 0$ such that $n > N_1$ implies $|x - x_n| < \dfrac{\varepsilon}{4}$. Likewise $\lim_{n\to\infty} y_n = y$ implies there is $N_2 > 0$ such that $n > N_2$ implies $|y_n - y| < \dfrac{\varepsilon}{6}$. Let $N = \max\{N_1, N_2\}$. Then if $n > N$ we have

$$\begin{aligned}|(2x_n + 3y_n) - (2x - 3y)| &= |2(x_n - x) + 3(y_n - y)| \\ &\le 2|x_n - x| + 3|y_n - y| \\ &< 2\left(\frac{\varepsilon}{4}\right) + 3\left(\frac{\varepsilon}{6}\right) \\ &= \varepsilon.\end{aligned}$$

That is $n > N$ implies $|(2x_n + 3y_n) - (2x - 3y)| < \varepsilon$. Therefore $\lim_{n\to\infty}(2x_n + 3y_n) = 2x + 3y$. □

2. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of real numbers with $\lim_{n\to\infty} x_n = 5$. Show that there is positive integer N such that $n > N$ implies $x_n < 6$. *Hint:* One way to do this is use $\varepsilon = 1$ in the definition of limit.

Solution: Let $\varepsilon = 1$ in the definition of $\lim_{n\to\infty} x_n = 5$. Then there is a $N > 0$ such that $n > N$ implies that

$$|x_n - 5| < \varepsilon = 1.$$

Then for $n > N$ we have $x_n \in B(5, 1) = (5 - 1, 5 + 1) = (4, 6)$

$$4 < x_n < 6$$

as required. □

3. (a) Define what it means for a sequence to be a ***Cauchy sequence***.

Solution: The sequence $\langle p_n\rangle_{n=1}^{\infty}$ in a metric space is ***Cauchy*** if and only if for all $\varepsilon > 0$ there is a positive integer N such that if $m, n > N$, then $d(p_m, p_n) < \varepsilon$. □

(b) Define what it means for a metric space to be ***complete***.

Solution: The metric space E is ***complete*** if and only if every Cauchy sequence in E converges to a point of E. $\square$

(c) Show that any Cauchy sequence in $\mathbb{R}$ is bounded.

Solution: Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Let $\varepsilon = 1$. Then by the definition of Cauchy there is a $N > 0$ such that if $m, n > N$ then $d(x_m, x_n) < \varepsilon = 1$. Let $a = N + 1$. Then we have $n > N$ implies $d(x_a, x_n) = |x_n - x_a| < 1$. That is $x_n \in B(x_a, 1)$ for all $n > N$. We still have to deal with the points $x_1, x_2, \ldots, x_N$. Let

$$r = 1 + \max\{|x_1 - x_a|, |x_2 - x_a|, \ldots, |x_N - x_a|\}.$$

Then $x_n \in B(x_a, r)$ for all n. (Or what is the same thing $x_a - r < x_n < x_a + r$ for all n.) $\square$

(d) Using that any sequence in $\mathbb{R}$ has a monotone subsequence and that every bounded monotone sequence is convergent, prove that the real numbers are complete.

Solution: Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. We need to show that this sequence converges to a point of $\mathbb{R}$. We know that this sequence will have a monotone subsequence, say $\langle x_{n_k} \rangle_{k=1}^{\infty}$. This subsequence is a bounded monotone sequence in $\mathbb{R}$ and we know that such sequences converge. Let x be its limit, that is $\lim_{k\to\infty} x_{n_k} = x$. We could now just quote the result that if a Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges and has the same limit as the subsequence.

But to be complete we prove directly that $\lim_{k\to\infty} x_{n_k} = x$ implies that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then there is a $N > 0$ such that $n, m > N$ implies that $|x_m - x_n| < \varepsilon/2$. As $\lim_{k\to\infty} x_{n_k} = x$ there is a $K > 0$ such that $k > K$ implies that $|x_{n_k} - x| < \varepsilon/2$. Now choose k so that $k > K$ and also $n_k > N$. Then for $n > N$ we have

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_k} + x_{n_k} - x_n| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - x_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

shows $\lim_{n\to\infty} x_n = x$ and completes the proof. $\square$

4. (a) Define what it means for the metric space E to be ***sequentially compact***.

Solution: The metric space E is ***sequentially compact*** if and only if every sequence in E has a subsequence that converges to a some point of E. $\square$

(b) Show that a sequentially compact space is complete. *Hint:* You can use the fact that if a Cauchy sequence has a convergent subsequence, then the original sequence converges.

Solution: Let E be sequentially compact and let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Then as E is sequentially compact there is a subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ that converges to some point p of E. As the original series, $\langle p_n \rangle_{n=1}^{\infty}$, is Cauchy this implies $\langle p_n \rangle_{n=1}^{\infty}$ also converges to p. This shows that every Cauchy sequence in E converges to a point of E and therefore E is complete. $\square$

5. (a) Define that is means for $\mathcal{U}$ to be an ***open cover*** of the set S.

Solution: $\mathcal{U}$ is an open cover of S if and only if each $U \in \mathcal{U}$ is an open set and for each $p \in S$ there is a $U \in \mathcal{U}$ with $p \in U$. $\square$

(b) Define what it means for the set S to be ***compact***.

Solution: The set S is ***compact*** if and only if every open cover of S has a finite subcover. $\square$

(c) Show that if S is a compact set that it can be covered by a finite number of open balls of radius 1.

Solution: Let S be compact. Let $=\{B(x,1) : x \in S\}$ be the collection of open balls centered at a point of S. This is an open cover of S as each $B(x,1)$ is an open set and if $x \in S$, then $x \in B(x,1) \in \mathcal{U}$. Because S is compact the open cover $\mathcal{U}$ has a finite subcover $\mathcal{U}_0 = \{B(x_1,1), B(x_2,1), \ldots, B(x_n,1)\} \subseteq \mathcal{U}$. The set $\mathcal{U}_0$ is a cover of S by a finite number of open balls of radius 1. $\square$