Mathematics 554H/703I Test 3 Name: Answer Key

1. Let $f\colon E \to E'$ be a map between metric spaces. State the precisely the theorem about f being continuous if and only if some condition about preimages of open sets holds.

Solution. The function $f\colon E \to E'$ is continuous if and only if for all open sets $U \subseteq E'$ the preimage $f^{-1}[U] = \{x \in E : f(x) \in U\}$ is an open subset of E. □

2. Let E be a compact metric space and $f\colon E \to \mathbb{R}$ a continuous function. Prove that the image of f is contained in some open interval $(-a, a)$.

Solution. For any positive real number a let U_a be the subset of E defined by

$$U_a = f^{-1}[(-a, a)] = \{x : f(x) \in (-a, a)\}.$$

The set $(-a, a)$ is open and f is continuous therefore U_a is an open subset of E. For each $x \in E$ there is a real number a with $|f(x)| < a$ (to be concrete note that $a = |f(x)| + 1$ works) and therefore $f(x) \in (-a, a)$, that is $x \in U_a$. Therefore $\mathcal{U} = \{U_a : a > 0\}$ is an open cover of E. But E is compact and therefore $\mathcal{U}$ has a finite subset $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}$ that covers E. Form the definition of U_a we see that if $a_i < a_j$ then $U_{a_i} \subseteq U_{a_j}$. Therefore if $a = \max\{a_1, a_2, \ldots, a_n\}$

$$E \subseteq U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n} = U_a.$$

But $E \subseteq U_a$ implies that $f(x) \in (-a, a)$ for all $x \in E$, that is the image of f is contained in $(-a, a)$. □

3. (a) Let E be a metric space and $f\colon E \to \mathbb{R}$ function. Define what it means for f to be ***continuous*** at the point $p_0 \in E$.

Solution. The funtion $f\colon E \to \mathbb{R}$ is continuous at the point p_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $p \in E$

$$d(p, p_0) < \delta \quad \text{implies} \quad |f(p) - f(p_0)| < \varepsilon.$$

□

(b) Show directly from the definition that if f is continuous at p_0, then so is its square f^2.

Solution. Let $g(p) = f(p)^2$. Assuming that f is continuous at p_0 we wish to show that g is also continuous at p_0. Let $\varepsilon > 0$. We start by noting that

$$(1) \quad |g(p) - g(p_0)| = |f(p)^2 - f(p_0)^2| = |f(p) + f(p_0)||f(p) - f(p_0)|.$$

Because f is continuous at p_0 there is a $\delta > 0$ such that

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |f(p) - f(p_0)| < 1.$$

But if $|f(p) - f(p_0)| < 1|f(p) - f(p_0)| < 1$ we have

$$\begin{aligned} |f(p) + f(p_0)| &= |2f(p_0) + (f(p) - f(p_0))| \\ &\leq 2|f(p_0) + |f(p) - f(p_0)| \\ &< 2|f(p_0)| + 1. \end{aligned}$$

Using this in the inequality (1) gives

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |g(p) - g(p_0)| < \big(2|f(p_0)| + 1\big)|f(p) - f(p_0)|.$$

Again using the continuity of f at p_0 we see that there is a $\delta_2 > 0$ such that

$$d(p, p_0) < \delta_2 \quad \text{implies} \quad |f(p) - f(p_0)| < \frac{\varepsilon}{2|f(p_0)| + 1}.$$

Therefore if we set $\delta = \min\{\delta_1, \delta_2\}$ we have that $d(p, p_0) < \delta$ implies

$$\begin{aligned} |g(p) - g(p_0)| &< \big(2|f(p_0)| + 1\big)|f(p) - f(p_0)| \\ &< \big(2|f(p_0)| + 1\big)\frac{\varepsilon}{2|f(p_0)| + 1} \\ &= \varepsilon. \end{aligned}$$

which is exactly what is needed to show that g is continuous at p_0. □

4. (a) Define what it means for the metric space E to be ***connected***.

Solution. The metric space E is connected if and only if it is not the disjoint union of two nonempty open subsets U and V.

Remark. When E is not connected, we have $E = U \cup V$ where U and V are nonempty open subsets of E and $U \cap V = \varnothing$. We call $E = U \cup V$ a ***disconnection*** of E. □

(b) Prove that if E is a connected metric space and $f\colon E \to \mathbb{R}$ is continuous, the the image of f is connected.

Solution. Towards a contradiction assume that $f[E]$ is not connected. Then $f[E] = U \cup V$ where U and V are nonempty open subsets of $f[E]$ and $U \cap V = \varnothing$. Because f is continuous the subsets $f^{-1}[U]$ and $f^{-1}[V]$ are open subsets of E. Also $f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V] = f^{-1}[\varnothing] = \varnothing$. Each of $f^{-1}[U]$ and $f^{-1}[V]$ is nonempty. Therefore

$E = f^{-1}[U] \cup f^{-1}[V]$ is a disconnection of E contradicting that E is connected. □

5. (a) State the Intermediate Value Theorem for a continuous function $f\colon [a,b] \to \mathbb{R}$.

Solution. Let y_0 be a number between $f(a)$ and $f(b)$. Then there is a $x_0 \in (a,b)$ with $f(x_0) = y_0$. □

(b) Prove the the polynomial $f(x) = x^4 + 2x - 8$ has at least two real roots, one positive and one negative.

Solution. The function f is continuous on all of $\mathbb{R}$ because it is a polynomial.

Note

$$f(1) = (1)^4 + 2(1) - 8 = -5 < 0$$
$$f(2) = (2)^4 + 2(2) - 8 = 12 > 0.$$

Therefore $y_0 = 0$ is between $f(1)$ and $f(2)$ so there is a number $x_0 \in (1,2)$ with $f(x_0) = 0$. This is a positive root of $f(x) = 0$.

Also

$$f(-1) = (-1)^4 + 2(-1) - 8 = -9 < 0$$
$$f(-2) = (-2)^4 + 2(-2) - 8 = 4 > 0.$$

Thus 0 is between $f(-2)$ and $f(-1)$ and thus there is a $x_1 \in (-2,-1)$ with $f(x_1) = 0$. This gives a negative root for $f(x) = 0$. □

6. Let E, E' and E'' be metric spaces and let $f\colon E \to E'$ and $g\colon E' \to E''$ be continuous functions. Prove that the composition $g \circ f$ is continuous. *Hint:* This can either be done with εs and δs or you can use the theorem stated in Problem 1.

Solution 1. Let $p_0 \in E$ and let $\varepsilon > 0$. Then, as g is continuous at $f(p_0)$ (it is continuous at all points), there is a $\delta_1 > 0$ such that

$$d'(q, f(p_0)) < \delta \quad \text{implies} \quad d''(q, g(f(p_0))) < \varepsilon.$$

Because f is continuous at p_0 there is a $\delta > 0$ such that

$$d(p, p_0) < \delta \quad \text{implies} \quad d'(f(p), f(p_0)) < \delta_1.$$

Thus if $d(p,p_0) < \delta$ we have that $d'(f(p), f(p_0)) < \delta$ and therefore

$$d(p, p_0) < \delta \quad \text{implies} \quad d''(g(f(p)), g(f(p_0))) < \varepsilon$$

which shows that the composition $g \circ f$ is continuous at p_0. As p_0 was ab arbitrary point of E this shows that $g \circ f$ is continuous on all of E. □

Solution 2. We use the fact that if $U \subseteq E''$ then

$$(g \circ f)^{-1}[U] = f^{-1}[g^{-1}[U]].$$

Now to show that $g \circ f$ is continuous, it is enough to show that for any open set $U \subseteq E''$ that $(g \circ f)^{-1}[U]$ is an open subset of E. But $g^{-1}[U]$ is open because g is continuous and U is open. But then $(g \circ f)^{-1}[U] = f^{-1}[g^{-1}[U]]$ is open because f is continuous and $g^{-1}[U]$ is open, which completes the proof. □