Mathematics 300

Quiz 19 Name: Key

You must show your work to get full credit.

Here are are going to derive some formulas that will be useful both in the remainder of this class and in your future math classes.

To start let x and y be any numbers and let n be a positive integer. Set

$$S_n = x^n + x^{n-1}y^n + x^{n-2}y^2 + \cdots + x^2y^{n-1} + y^n$$

Thus

$$\begin{aligned} S_1 &= x + y \\ S_2 &= x^2 + xy + y^2 \\ S_3 &= x^3 + x^2y + xy^2 + y^3 \\ S_4 &= x^4 + x^3y + x^2y^2 + xy^3 + y^4 \end{aligned}$$

Consider what happens if we multiple one of these, say S_3 by x or y:

$$\begin{aligned} xS_3 &= x^4 + x^3y + x^2y^2 + xy^3 \\ yS_3 &= \phantom{x^4 + {}} x^3y + x^2y^2 + xy^3 + y^4 \end{aligned}$$

What is $(x-y)S_5 =$ $x^4 - y^4$

What is xS_5? $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5$

What is yS_5? $x^5y + x^4y^2 + x^3y^2 + x^2y^4 + xy^5$

What is $(x-y)S_5$? $x^6 - y^6$

We can now do the general case:

1. What is xS_n?

$x^{n+1} + x^ny + x^{n-1}y^{n-1} + \cdots + x^2y^{n-1} + xy^n$

2. What is yS_n? $x^ny + x^{n-1}y^{n-1} + \cdots + x^2y^{n-1} + xy^n + y^{n-1}$

3. What is $(x-y)S_n$? $= x^{n+1} - y^{n+1}$

4. Letting $m = n + 1$ give a formula for factoring $x^m - y^m$.

$$x^m - y^m = \underline{(x-y)(x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \cdots + x\, y^{n-2} + y^{n-1})}$$

Let us put these formulas to work to prove a few things.

5. Show that if n and k are positive integers, and $x, y \in \mathbb{Z}$ with

$$x \equiv y \pmod{n}.$$

Show that

$$x^k \equiv y^k \pmod{n}.$$

<u>Proof</u> We are given that $x \equiv y \bmod n$. Thus

$$x - y = q n$$

for some integer q. So (use our formula above)

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \quad + xy^{k-2} + y^{k-1})$$
$$= qn(x^{k-1} + x^{k-2}y + \quad + xy^{k-2} + y^{k-1})$$
$$= q'n$$

where $q' = q(x^{k-1} + x^{k-2}y + \quad + xy^{k-2} + y^{k-1})$ is an integer. Thus $n \mid x^k - y^k$. So $x^k \equiv y^k \bmod n$

As a particular case note this implies that

$$10^k \equiv 1^k \equiv 1 \pmod{9}.$$

6. Use this to show explain why the congruence

$$2{,}135 \equiv 2 + 1 + 3 + 5 \pmod{9}$$

holds. write

$$2{,}135 = 2(10)^3 + 1\cdot(10)^2 + 3(10) + 5$$
$$\equiv 2(1)^3 + 1\cdot(1)^2 + 3(1) + 5 \quad \bmod 9$$
$$\equiv 2 + 1 + 3 + 5 \quad \bmod 9$$
$$= 11$$
$$\equiv 2 \quad \bmod n.$$