Mathematics 300

Quiz 34 Name: Key

You must show your work to get full credit.

1. Write a sentence or two explaining why 32 is the sum of two prime numbers.

The numbers 3 and 29 are prime and $32 = 3 + 29$

2. Use the $10 \equiv -1 \pmod 11$ to explain why

$$4{,}538 \equiv -4 + 5 - 3 + 8 \pmod{11}$$

Recall that $4{,}538 = 4(10)^3 + 5(10)^2 + 3(10) + 8$. So

$$\begin{aligned} 4{,}538 &= 4(10)^3 + 5(10)^2 + 3(10) + 8 \\ &\equiv 4(-1)^3 + 5(-1)^2 + 3(-1) + 8 \pmod{11} \\ &\equiv -4 + 5 - 3 + 8 \pmod{11} \end{aligned}$$

3. (a) The rational root test for cubic polynomials is that it $r = \dfrac{p}{q}$ is a rational root in lowest terms of

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

where a_0, a_1, a_2, a_3 are integers then $p \mid a_0$ and $q \mid a_3$. Use this to explain why

$$x^3 - 11$$

has no rational roots.

Towards a contradiction assume that the equation $x^3 - 11$ has a rational root $r = p/q$ in lowest terms. Then $p \mid 11$, so $p = \pm 1$ or $p = \pm 11$ and $q \mid 1$ so $q = \pm 1$. Thus $r = \pm 11$ or $r = \pm 1$.

But
$$\begin{aligned} (1)^3 - 11 &= -10 \neq 0 \\ (-1)^3 - 11 &= -12 \neq 0 \\ (11)^3 - 11 &= 1331 - 11 \neq 0 \\ (-11)^3 - 11 &= -1331 - 11 \neq 0. \end{aligned}$$

This is a contradiction so there are no rational roots.

(b) Show that $\sqrt[3]{11}$ is irrational.

The number $\sqrt[3]{11}$ is a root of $x^3-11=0$. We saw in part (a) that this equation has no rational roots. Thus $\sqrt[3]{11}$ is irrational.

4. Prove or give a disproof. The cube of an irrational numbers is irrational. False.

Let $\alpha = \sqrt[3]{11}$. We have just shown that α is irrational. But $\alpha^3 = 11$ is rational.

5. Prove or give a disproof. There are integers x and y with $7x - 14y = 3$. False.

If such integer existed we would have

$7(x-2y) = 7x-14y=3$. This implies $7 \mid 3$ which is a contradiction.

6. Prove or give a disproof. There are integers x and y with $7x - 14y = 7$. True

Let $x=3$ and $y=1$. Then x and y are integers and

$$
\begin{aligned}
7x - 14y &= 7(3) - 14(1) \\
&= 21 - 14 \\
&= 7.
\end{aligned}
$$

7. Use induction to show that for every integers $n \geq 1$, that $3 \mid (n^3 - n)$. *Hint:* You may want to use the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

The base case is $n=1$. $1^3 - 1 = 0$ and $3 \mid 0$. so this holds.

Induction hypothesis $3 \mid k^3 - k$. That is $k^3 - k = 3q$ for some $q \in \mathbb{Z}$.

Then $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$
$= (k^3 - k) + 3(k^2 + k)$
$= 3q + 3(k^2 + k) = 3(q + k^2 + k)$.

$q + k^2 + k \in \mathbb{Z}$ so this implies $3 \mid (k+1)^3 - (k+1)$ and finishing the induction.

8. If $g(n) = -2g(n-1)$ and $g(0) = 5$, show that $g(n) = 5(-2)^n$ for all $n \geq 1$.

We use induction.

Base case $g(0) = 5(-2)^0 = 5(1) = 5$. so this holds

Induction hypothesis $g(k) = 5(-2)^k$. Then

$g(k+1) = (-2)g(k) = (-2)5(-2)^k = 5(-2)^{k+1}$.

This shows the induction conclusion holds and finishes the proof.

9. Find the first four derivatives of $f(x) = e^{3x}$. Then guess a formula for $f^{(n)}(x)$ and use induction to prove your guess is correct.

$f'(x) = 3e^{3x}$, $f''(x) = 3^2 e^{3x}$, $f'''(x) = 3^3 e^{3x}$, $f^{(4)}(x) = 3^4 e^{3x}$

so we guess that $f^{(n)}(x) = 3^n e^{3x}$.

The base case is $f^{(1)}(x) = f'(x) = 3^1 e^{3x} = 3e^{3x}$ and this holds.

Induction hypothesis $f^{(k)}(x) = 3^k e^{3x}$. Then

$f^{(k+1)}(x) = (f^{(k)}(x))' = (3^k e^{3x})'$
$= 3^k \, 3 e^{3x}$
$= 3^{k+1} e^{3x}$

which is the induction conclusion. done

10. (a) Define R is a ***relation*** on the set A.

This means $R \subseteq A \times A$.

(b) Define the relation R is ***symmetric***.

If xRy, then yRx for all $x, y \in A$.

(c) Define the relation R is ***reflective***.

xRx for all $x \in A$.

(d) Define the relation R is ***transitive***.

If xRy and yRz, then xRz for all $x, y, z \in A$.

11. Let on the set of real numbers let R be the relations

$$xRy \iff |x - y| < 4.$$

(a) Is R symmetric? (Prove your result.)

Yes. If xRy, then $|x-y| < 4$. But $|y-x| = |x-y|$ so $|y-x| = |x-y| < 4$. So yRx holds.

(b) Is R reflective? (Prove your result.)

Yes. If $x \in \mathbb{R}$, then $xRx \Leftrightarrow |x-x| = 0 < 4$. This is true.

(c) Is R transitive? (Prove your result.) NO.

Let $x=0$, $y=3$, $z=6$. Then $|x-y| = 3 < 4$ so xRy. and $|y-z| = 3 < 4$ so $3R6$. But $|x-z| = 6 \not< 4$ so $x \not R z$.