Mathematics 300

Quiz 37 **Name:** Key

You must show your work to get full credit.

1. (a) List the elements of the set $\{n \in \mathbb{Z} : n^2 - 5 < 0\}$ between brackets.

$\underline{\{-2, -1, 0, 1, 2\}}$

2. (a) If A is a set, define the ***power set*** of A.

$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$

(b) Let $A = \{a, b\}$ and $B = \{1\}$. Then what are the following:

$A \times B = \underline{\{(a,1),(b,1)\}}$

$\mathcal{P}(A) = \underline{\{\varnothing, \{a\}, \{b\}, (a,b)\}}$

3. What is the negation of the sentence: For each positive number ε, there is a positive integer N such that for all $n \geq N$ the inequality $|a_n - a| < \varepsilon$ holds.

Solution: There exists a positive number $\varepsilon > 0$ such that for all positive integers N, there is a $n \geq N$ such that $|a_n - a| \geq \varepsilon$. □

4. Give a contrapositive proof that if a^4 is even, then a is even.

Solution: The contrapositive is: If a is odd then a^4 is odd. Assume that a is odd. Then $a \equiv 1 \pmod 2$. Therefore

$$\begin{aligned} a^4 &\equiv 1^4 \pmod 2 \\ &\equiv 1 \pmod 2. \end{aligned}$$

This implies that a^4 is odd which finishes the proof. □

5. Use the last problem to give a proof by contradiction that $\sqrt[4]{2}$ is not a rational number.

Solution: Towards a contradiction that $\sqrt[4]{2}$ is a rational number. Then we can write

$$\sqrt[4]{2} = \frac{p}{q}$$

where p and q are integers, $q \neq 0$ and the fraction is in lowest terms. Rise both sides of this last equation to the fourth poser and multiple by q^4 to get

$$2q^4 = p^4.$$

This implies that $2 \mid p^4$, that is p^4 is even. Then by the previous problem this implies that p is even. That is $p = 2a$ for some integers a. Use this in the equation $2q^4 = p^4$ to get:

$$\begin{aligned} 2q^4 &= (2a)^4 \\ q^4 &= 8a^4 && \text{(divide by 2)} \\ q^4 &= 2(4a^4) \end{aligned}$$

This implies that q^4 is even. So using the previous problem again we find that q is even. That is $q = 2b$ for some integer b. But then we have

$$\frac{p}{q} = \frac{2a}{2b}$$

which contradicts our assumption the fraction $\dfrac{p}{q}$ was in lowest terms. □

6. Use proof by cases to show that for all integers n that $3 \mid (n^3 + 2n)$.

Solution: There are three cases.

Case 1. $n \equiv 0 \pmod 3$. Then

$$\begin{aligned} n^3 + 2n &\equiv 0^3 + 2(0) \pmod 3 \\ &\equiv 0 \pmod 3. \end{aligned}$$

And $n^3 + 2n \equiv 0 \pmod 3$ implies that $3 \mid (n^3 + 2n)$ in this case.

Case 2. $n \equiv 1 \pmod 3$. Then

$$\begin{aligned} n^3 + 2n &\equiv 1^3 + 2(1) \pmod 3 \\ &\equiv 3 \pmod 3 \\ &\equiv 0 \pmod 3. \end{aligned}$$

And $n^3 + 2n \equiv 0 \pmod 3$ implies that $3 \mid (n^3 + 2n)$ in this case.

Case 3. $n \equiv 1 \pmod 3$. Then

$$\begin{aligned} n^3 + 2n &\equiv 2^3 + 2(2) \pmod 3 \\ &\equiv 12 \pmod 3 \\ &\equiv 0 \pmod 3. \end{aligned}$$

And $n^3 + 2n \equiv 0 \pmod 3$ implies that $3 \mid (n^3 + 2n)$ in this case.

Thus $3 \mid (n^3 + 2n)$ in all cases, which finishes the proof. □

7. Let $A = \{n \in \mathbb{Z} : 6 \mid n\}$ and $B = \{12x + 18y : x, y \in \mathbb{Z}\}$. Prove that $A = B$.

Solution: We need to prove the two inclusions $A \subseteq B$ and $B \subseteq A$.

Proof that $A \subseteq B$. Let $a \in A$. Then by the definition of A we have that $6 \mid a$. That is for some integers q we the equality $b = 6q$. Therefore

$$\begin{aligned} a &= 6q \\ &= (-12 + 18)q \\ &= 12(-q) + 18q \\ &= 12x + 18y \end{aligned}$$

where $x = -q$ and $y = q$ are integers. Therefore $a \in B$. This shows that $A \subseteq B$.

Proof that $B \subseteq A$. Let $b \in B$. Then $b = 12x + 18y$ for some integers x and y. Then

$$b = 12x + 18y = 6(2x + 3y) = 6q$$

where q $2x + 3y$ is an integer. Therefore $6 \mid b$. This shows that $b \in A$ and completes the proof that $B \subseteq A$. □

8. (a) Defined d is the ***greatest common divisor*** of the positive integers a and b.

Solution: $d = \gcd(a, b)$ if d is the largest integer that divides both a and b. □

(b) Prove that if $5a^3 - 4b^3 = 1$ that $\gcd(a, b) = 1$.

Solution: Let d be any integer that divides both a and b. That is $d \mid a$ and $d \mid b$. Then there are integers m and n such that

$$a = md \qquad \text{and} \qquad b = nd.$$

Use these equation in $5a^3 - 4b^3 = 1$ to get

$$5(md)^3 - 4(nd)^3 = 1.$$

Factor out a d to get

$$d(5m^3d^2 - 4n^3d^2) = 1.$$

As $5m^3d^2 - 4n^3d^2$ is an integer this implies $d \mid 1$. But the only integers that divide 1 are 1 and -1. That is the only integers that divide both a and b are 1 and -1 and the largest of these is 1. Therefore $\gcd(a, b) = 1$. $\square$

9. (a) Prove or disprove: The sum of two rational numbers is rational.

Solution: This is true. Let r and s be rational numbers. Then there are integers a, b, c, d with $b \neq 0$ and $d \neq 0$ such that

$$r = \frac{a}{b} \qquad \text{and} \qquad \frac{c}{d}.$$

We now add r and s:

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bd}{bd} = \frac{p}{q}$$

where $p = ad + bc$ and $q = bd$ are integers and $q \neq 0$. Thus the sum $r + s$ is rational. $\square$

(b) Prove or disprove the sum of two negative irrational numbers is irrational.

Solution: This is false. We know that the numbers $\sqrt{2}$ is irrational. From this it is not hard to check that the numbers

$$a = -\sqrt{2} \qquad \text{and} \qquad b = -3 + \sqrt{2}$$

are irrational. And they are both negative. But their sum is

$$a + b = -\sqrt{2} + (-3 + \sqrt{2}) = -3$$

and -3 is rational. $\square$

10. Let $f(n)$ be a function defined on the non-negative integers such that

(1) $$f(0) = 0 \qquad \text{and} \qquad f(n) = \frac{f(n-1) + 1}{2}.$$

Prove that for $n \geq 0$ that

$$f(n) = \frac{2^n - 1}{2^n}$$

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Solution: **Base case:** This is $n = 0$.

$$\frac{2^0 - 1}{2^0} = \frac{1 - 1}{1} = 0 = f(0).$$

Therefore the base case holds.

Induction hypothesis: $f(k) = 2^k - 12^k - 1$. We now use the equation (1) with $n = k + 1$ to get

$$\begin{aligned} f(k+1) &= \frac{f(k) + 1}{2.} \\ &= \frac{\frac{2^k - 1}{2^k} + 1}{2} \\ &= \frac{\frac{2^k - 1 + 2^k}{2^k}}{2} \\ &= \frac{\frac{2 \cdot 2^k - 1}{2^k}}{2} \\ &= \frac{2^{k+1} - 1}{2^{k+1}} \end{aligned}$$

That is

$$f(k+1) = \frac{2^{k+1} - 1}{2^{k+1}}$$

which is the induction conclusion. This completes the proof. $\square$