Mathematics 300 Homework, September 20, 2017.

You should memorize the definitions 4.1 (even integer) 4.2 (odd integer) and 4.3 (same and opposite parity) on page 89 of the text. I will ask this on the quiz on Friday.

On page 100 do problems 1, 3, and 5.

Here are some more problems.

1. If x is an odd integer, then $x^2 + 5$ is even.
2. If x and y are odd integers, then $x^2 + y^2$ is even.
3. If a is an odd integer and then $a^3 + 4$ is also odd.
4. The sum of any three odd numbers is odd.

And here is a problem to be written up and handed in.

5. If a is an odd integer and b is an even integer, then $(a+3)(b-1)$ is even.

Solutions to Problems 1–4 on the next two pages.

Here are a couple of formulas you should know from algebra and which will come up several times in coming weeks.

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.$$

As examples

$$\begin{aligned}(2n+1)^3 &= (2n)^3 + 3(2n)^2 + 3(2n) + 1\\ &= 8n^3 + 12n^2 + 4n + 1.\end{aligned}$$

(Here we had $x = 2n$ and $y = 1$.

$$\begin{aligned}(3a-2b)^3 &= (3a)^3 - 3(3a)^2(2b) + 3(3a)(2b)^2 - (2b)^3\\ &= 27a^3 - 54a^2b + 36ab^2 - 8b^3\end{aligned}$$

(This time $x = 3a$ and $y = 2b$.)

Proof for Problem 1. If x is an odd integer, then $x = 2a+1$ for some integer a. Thus

$$\begin{aligned} x^2 + 5 &= (2a+1)^2 + 5 && \text{(as } x = 2a+1\text{)} \\ &= 4a^2 + 4a + 1 + 5 \\ &= 4a^2 + 4a + 6 \\ &= 2(2a^2 + 2a + 3) \\ &= 2b \end{aligned}$$

where $b = 2a^2 + 2a + 1 \in \mathbb{Z}$ is an integer. Therefore $x^2 + 5$ is even. □

Proof for Problem 2. If x and y are odd integers, then

$$\begin{aligned} x &= 2a + 1 \\ y &= 2b + 1 \end{aligned}$$

for some integers $a, b \in \mathbb{Z}$. Using these formulas for x and y in $x^2 + y^2$ gives

$$\begin{aligned} x^2 + y^2 &= (2a+1)^2 + (2b+1)^2 \\ &= 4a^2 + 4a + 1 + 4b^2 + 4b + 1 \\ &= 4a^2 + 4a + 4b^2 + 4b + 2 \\ &= 2(2a^2 + 2a + 2b^2 + 2b + 1) && = 2c \end{aligned}$$

where $c = 2a^2 + 2a + 2b^2 + 2b + 1 \in \mathbb{Z}$ is an integer. Thus $x^2 + y^2$ is even. □

Proof for Problem 3. If a is an odd integer then

$$a = 2k + 1$$

for some integer $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} a^3 + 4 &= (2k+1)^3 + 4 \\ &= (2k)^3 + 3(2k)^2 + 3(2k) + 1 + 4 \\ &= 8k^3 + 12k^2 + 6k + 5 \\ &= 8k^3 + 12k^2 + 6k + 4 + 1 \\ &= 2(4k^3 + 6k^2 + 3k + 2) + 1 \\ &= 2b + 1 \end{aligned}$$

where $b = 4k^3 + 6k^2 + 3k + 2 \in \mathbb{Z}$ is in integer. Thus $a^3 + 4$ is odd. □

Proof for Problem 4. Let a, b, and c be any odd numbers. We wish to show that $a+b+c$ is odd. By the definition of odd

$$a = 2x + 1$$
$$b = 2y + 1$$
$$c = 2z + 1$$

for some integers $x, y, z \in \mathbb{Z}$. Then

$$\begin{aligned} a + b + c &= 2x + 1 + 2y + 1 + 2z + 1 \\ &= 2x + 2y + 2z + 3 \\ &= 2x + 2y + 2z + 2 + 1 \\ &= 2(x + y + z + 1) + 1 \\ &= 2m + 1 \end{aligned}$$

where $m = x + y + 1 \in \mathbb{Z}$ is an integer. Therefore $a + b + c$ is odd. $\square$