Mathematics 300 Homework, October 2, 2017.

1. Prove that for any integer n that $n^3 - n$ is divisible by 3.

2. Show that for any real numbers a and b that $(a+b)^3 = a^3 + b^3$, then $a = 0$, $b = 0$ or $a = -b$.

Read Pages 102–105 about contrapositive proofs and make sure you understand the proofs on these pages. Then do Problems 1–9 odd.

The Solutions to the first two problems are on the next page.

Solution to Problem 1. Let r be the remainder if n is divided by 3. That is $n = 3q + r$ where $0 \le r < 3$. Thus $r = 0, 1, 2$. There are three cases

Case 1. $r = 0$. Then $n = 3q$ and

$$\begin{aligned} n^3 - n &= (3q)^3 - 3q \\ &= 3(9q^3 - q) \\ &= 3k \end{aligned}$$

where $k = 9q^3 - q \in \mathbb{Z}$ and thus $3 \mid n^3 - n$.

Case 2. $r = 1$. Then $n = 3q + 1$ and thus

$$\begin{aligned} n^3 - n &= (3q+1)^3 - (3q+1) \\ &= (3q)^3 + 3(3q)^2 + 3(3q) + 1 - 3q - 1 \\ &= 27q^3 + 27q^2 + 6q \\ &= 3(9q^3 + 9q^2 + 2q) \\ &= 3k \end{aligned}$$

where $k = 9q^3 + 9q^2 + 2q$ is an integer. Thus $3 \mid n^3 - n$.

Case 3. $r = 2$. Then $n = 3q + 2$ and we have

$$\begin{aligned} n^3 - n &= (3q+2)^3 - (3q+2) \\ &= (3q)^3 + 3(3q)^2(2) + 3(3q)(2)^2 + (2)^3 - 3q - 2 \\ &= 27q^3 + 54q^2 + 15q + 6 \\ &= 3(9q^3 + 18q^2 + 5q + 2) \\ &= 3k \end{aligned}$$

where $k = 9q^3 + 18q^2 + 5q + 2 \in \mathbb{Z}$. Thus $3 \mid n^3 - n$. □

Solution to Problem 2. Assume that $(a + b)^3 = a^3 + b^3$. Then

$$\begin{aligned} 0 &= (a+b)^3 - a^3 - b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3 \\ &= 3a^2b + 3ab^2 \\ &= 3ab(a+b). \end{aligned}$$

Dividing by 3 gives

$$ab(a + b) = 0.$$

The only way a product of three numbers can be zero is that if one of them is zero. Thus we have $a = 0$, or $b = 0$, or $(a + b) = 0$. But $(a + b) = 0$ is equivalent to $a = -b$. □