Mathematics 300 Homework, October 11, 2017.

On page 110 do problems 1, 3, 11 and the problems below. I will collect problem 3 below.

Recall that a number r is a ***rational number*** if

$$r = \frac{a}{b}$$

where a and b are integers and $b \neq 0$. We denote the set of rational numbers by $\mathbb{Q}$.

We now verify some of the basic properties of rational numbers.

Proposition 1. *The sum of two rational numbers is a rational numbers.*

Proof. We did this in class, so see your class notes. □

Proposition 2. *The difference of two rational numbers is a rational number.*

Problem 1. Prove this. □

Proposition 3. *Let r be a rational number with $r \neq 3$. Then*

$$s = \frac{2+r}{r-3}$$

is also rational.

Problem 2. Prove this. □

Proposition 4. *If r is a rational number and $r \neq 1$. Then*

$$s = \frac{r^3 - 4r + 1}{r-1}$$

is also a rational number.

Problem 3. Prove this.

Proof of Proposition 2. Let r and s be rational numbers. Then there are integers $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{aligned} r &= \frac{a}{b} \\ s &= \frac{c}{d} \end{aligned}$$

and $b, d \neq 0$. Then the difference of r and s is

$$\begin{aligned} r - s &= \frac{a}{b} - \frac{c}{d} \\ &= \frac{ad - bc}{bd} \\ &= \frac{p}{q} \end{aligned}$$

where $p = ad - bc$ and $q = bd$ are integers. Also $q = bd \neq 0$ as b and d are both not equal to zero. Thus the difference $r - s$ is a rational number. □

Proof of Proposition 3. As r is a rational number, by definition we have

$$r = \frac{a}{b}$$

where $a, b \in \mathbb{Z}$ and $b \neq 0$. We are also given that $r \neq 3$. This implies that $a \neq 3b$. We now have

$$\begin{aligned} s &= \frac{2 + r}{r - 3} \\ &= \frac{2 + \frac{a}{b}}{\frac{a}{b} - 3} \\ &= \frac{\left(2 + \frac{a}{b}\right) b}{\left(\frac{a}{b} - 3\right) b} && \text{(multiply top and bottom by } b\text{)} \\ &= \frac{2b + a}{a - 3b} \\ &= \frac{p}{q} \end{aligned}$$

where $p = 2b + a$ and $q = a - 3b$ are integers and $q \neq 0$ (as $a \neq 3b$). Therefore s is a rational number. □