Mathematics 300 Homework, October 16.

For the rest of the term you will be expected to know the formulas

$$\begin{aligned} x^2 - y^2 &= (x-y)(x+y) \\ x^3 - y^3 &= (x-y)(x^2+xy+y^2) \\ x^4 - y^4 &= (x-y)(x^3+x^2y+xy^2+y^3) \\ x^5 - y^5 &= (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) \end{aligned}$$

and in general

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

Here are a few problems for practice.

1. Show that $x^3 - 27$ factors.

Solution: Write this as $x^3 - 3^3$ and then we have

$$\begin{aligned} x^3 - 27 = x^3 - 3^3 \\ &= (x-3)(x^2 + x(3) + (3)^2 \\ &= (x-3)(x^2+3x+9). \end{aligned}$$

□

2. Show $a^5 + b^5$ factors.

Solution: Because of the plus sign this does not fit exactly in to the $x^n - y^n$ pattern. But just a little bit of trickery fixes this. Use that $(-1)^5 = -1$ to write $a^5 + b^5 = a^5 - (-b)^5$ and therefore

$$\begin{aligned} a^5 + b^5 &= a^5 - (-b)^5 \\ &= (a-(-b))(a^4 + a^3(-b) + a^2(-b)^2 + a(-b)^3 + (-b)^4 \\ &= (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4). \end{aligned}$$

□

3. Show that for all positive integers n that $9^n - 2^n$ is divisible by 7.

Solution: We use our $x^n - y^n$ identity again.

$$\begin{aligned} 9^n - 2^n &= (9-2)(9^{n-1} + 9^{n-2}2 + \cdots + 9^2 2^{n-2} + 9 \cdot 2^{n-2} + 2^{n-1}) \\ &= 7q \end{aligned}$$

where

$$q = 9^{n-1} + 9^{n-2}2 + \cdots + 9^2 2^{n-2} + 9 \cdot 2^{n-2} + 2^{n-1}$$

and $q \in \mathbb{Z}$. Thus $7 \big| (9^n - 2^n)$. □

Recall that the ***absolute value*** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

4. Show that for all $x \in \mathbb{R}$ that $|x| \geq 0$.

Solution: There are two cases $x \geq 0$ and $x < 0$.

Case 1. $x \geq 0$. In this case $|x|x \geq 0$ as required.

Case 2 $x < 0$. That is x is negative. Then $|x| = -x$ and the negative of a negative number is positive and so in this case $|x| = -x > 0$. □

5. Show that for all $x, y \in \mathbb{R}$ that $|xy| = |x||y|$.

Solution: This is annoying in that the natural proof splits into two many cases.

We first note that $|0| = 0$. Also any number multiplied by 0 is 0. Therefore is either x or y is 0, then both of $|xy|$ and $|x||y|$ are zero and therefore the required equality holds if either x or y is zero. We therefore only need to consider case where both x and y are non-zero.

Case 1. $x > 0$ and $y > 0$. Then from basic properties of inequalities (that is positive times positive is positive) $xy > 0$. Therefore

$$|xy| = xy = |x||y|.$$

Case 2. $x > 0$ and $y < 0$. Then from properties of inequalities (this time positive times negative is negative) $xy < 0$. Therefore

$$|xy| = -(xy) = x(-y) = |x||y|.$$

Case 3. $x < 0$ and $y > 0$. Then (this time using negative times positive is negative) we have $xy < 0$ and so

$$|xy| = -(xy) = (-x)(y) = |x||y|.$$

Case 3. $x < 0$ and $y < 0$. Then (negative times negative is positive) we have $xy > 0$ and thus

$$|xy| = xy = (-x)(-y) = |x||y|.$$

This covers all cases and completes the proof. □

Proposition 1. *If r and s are real numbers and $n > 0$. Assume*

$$0 \leq r, s < n.$$

Then

$$|r - s| < n.$$

(Here we are using the notation $0 \leq r, s < n$ as short hand for saying that $0 \leq r < n$ and $0 < s < n$ are both true.)

Proof. There are two cases.

Case 1. $r \leq s$. Then $s - r \geq 0$ (which is the same as $r - s < 0$ and therefore

$$|r - s| = -(r - s) = s - r.$$

Then

$$\begin{aligned} |r-s| &= s-r \\ &< n-r && \text{(as } s<n\text{)} \\ &\le n && \text{(as } r\ge 0\text{)} \end{aligned}$$

And therefore $|r-s|<n$ in this case. □

Case 2. $s<r$. **Doing this case is homework to be handed in.**