Mathematics 300 Homework, October 25, 2017.

Here are some examples of showing existence results.

1. Show that there exist integers x and y such that $x^2 + y^2 = 25$.

2. Show that every even integer is the sum of two odd integers.

3. Show that every even number between 20 and 30 inclusive is the sum of two prime numbers.

Read Chapter 8 up to Section 8.4 (that is pages 131–139). On Page 145 do Problems 1, 3, 7, and 27.

Solution to Problem 1. Let $x = 3$ and $y = 4$, then $x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25$. This shows the existence of integers x and y with $x^2 + y^2 = 25$. □

Solution to Problem 2. Before starting the proof let us look at some examples:

$$\begin{aligned} 2 &= 1 + 1 \\ 4 &= 3 + 1 \\ 6 &= 5 + 1 \\ 8 &= 7 + 1 \\ 10 &= 9 + 1 \end{aligned}$$

Let us now do the proof: Let a be an even integer. Then $a = 2k$ for some integer k Now write

$$a = 2k = (2k - 1) + 1.$$

This shows that a is the sum of the odd numbers $2k - 1$ and 1. □

Solution to Problem 3. We just exhibit each even number n with $20 \leq n \leq 30$ as a sum of two primes.

$$\begin{aligned} 20 &= 3 + 17 \\ 22 &= 3 + 19 \\ 24 &= 5 + 19 \\ 26 &= 3 + 23 \\ 28 &= 5 + 23 \\ 30 &= 7 + 23. \end{aligned}$$

One of the most famous unsolved problems in mathematics is to show that every even number greater than 4 is the sum of two primes. This was conjectured by the German mathematician Christian Goldbach in 1742. In the 275 years since then many people have worked on this, but it unproved. It is know that even even number greater than 4 but less than 4,000,000,000,000,000,000 is the sum of two primes. □