Mathematics 300 Homework, October 28, 2017.

We have a test on this coming Wednesday. As usual study by looking at the old homework and quizzes. Here are a few things you should have memorized.

- The definitions from Test 1.
- The definition of the ***greatest common divisor*** of two integers.
- The ***division algorithm***: Let a and b be integers with $b > 0$. Then there are unique integers q (the quotient) and r (the remainder) such that
$$a = qb + r \qquad \text{and} \qquad 0 \le r < b.$$
- If a, b, and n are integers with n positive know the definition of $a \equiv b \pmod{n}$.
- Know the statement of ***Bézout's Theorem***: If a and b are positive integers, then there are integers x_0 and y_0 such that
$$ax_0 + by_0 = \gcd(a, b).$$

Here are some problems that will be close to ones that will be on the test.

Let a and b be positive integers and let
$$A = \{ax + by : x, y \in \mathbb{Z}\}.$$
Define d to be
$$d = \text{Smallest positive element of } A.$$
Because $d \in A$ there are integers x_0 and y_0 such that
$$d = ax_0 + by_0.$$

1. Show that $d\big|b$. *Hint:* Towards a contradiction assume that d does not divide b. Then by the division algorithm there are integers q and r with
$$b = qd + r \qquad \text{and} \qquad 0 < r < d.$$
The reason that $0 < r$ (rather than $0 \le r$) is that we are assuming that d does not divide evenly into b. Now use $d = ax_0 + by_0$ in $b = qd + r$ and do some algebra so show that
$$r = ax + by$$
where x and y are integers. (You should give formulas for x and y.) This shows that $r \in A$. Explain (and this means using at least one English sentence) why this gives a contradiction.

2. Now do a similar argument to show that d divides a. (We did this in class on Friday.)

3. Use that $d = ax_0 + by_0$ to show
$$\gcd(a, b)\big|d.$$

4. Put the previous problems together to show that $d = \gcd(a, b)$.