Mathematics 300 Homework, November 13, 2017.

On page 170 of the text do problems 13 and 17.

1. What postages for a letter are possible using only 5¢ and 7¢ stamps? Prove your answer.

2. For all integers $n \geq 0$ let $g(n)$ satisfy

$$g(n) = 3g(n-1) - 4 \qquad \text{and} \qquad g(0) = 3.$$

Prove that $g(n) = 3^n + 2$ for all positive integers n.

Solution to Problem 1. We start by making a table for small values of n so see which can be expressed as a sum of nothing but 5's and 7's.

n¢	Is postage possible?
1	No.
2	No.
3	No.
4	No.
5	Yes: 5
6	No.
7	Yes: 7
8	No.
9	No.
10	Yes: $5+5$
11	No.
12	Yes: $5+7$
13	No.
14	Yes: $7+7$
15	Yes: $5+5+5$
16	No.
17	Yes: $5+5+7$
18	No.
19	Yes: $5+7+7$
20	Yes: $5+5+5+5$
21	Yes: $7+7+7$
22	Yes: $5+5+5+7$
23	No.
24	Yes: $5+5+7+7$
25	Yes: $5+5+5+5$
26	Yes: $5+7+7+7$
27	Yes: $5+5+5+5+7$
28	Yes: $7+7+7+7$
29	Yes: $5+5+5+7+7$
30	Yes: $5+5+5+5+5+5$

So it looks like we can do every postage other than

$$1, 2, 3, 4, 6, 8, 9, 13, 16, 18, 23.$$

We now prove this. For n¢ with $n \leq 22$ this follows from our table. So we are done if we can show

If $n \geq 30$, we can nake n¢ postage using 5 and 7 cent stamps.

In this case the base case $n = 30$ and as 30 can be expressed as sum of six 5's the base case holds.

The induction hypothesis is that we can make k¢ postage using 5 and 7 sent stamps.

If in making the k¢ postage we used two or more 7's, then take out two 7's and add in three 5's to get $k - 2(7) + 3(5) = k + 1$¢ and we are done.

If there are at most one 7's then, as $k \geq 30$, there must be at least four 5's. (otherwise the sum is at most $3(5) + 7 = 22$). So take out four 5's and add in three 7's to get $k - 4(5) + 3(7) = k + 1$¢.

So if $k \geq 30$ and we can put k¢ on a letter, then we can put $(k + 1)$¢ on a letter. This closes the induction and finishes the proof. □

Solution to Problem 2. Here the base case is $n = 0$. In that case we have

$$3^0 + 2 = 1 + 2 = 3 = g(0)$$

and so the base case holds.

The induction hypothesis is S_k: the formula $g(k) = 3^k + 2$ holds. (Our goal is to prove that S_{k+1}: the formula $g(k + 1) = 3^{k+1} + 2$ holds.) Letting $n = k + 1$ in $g(n) = 3g(n - 1) - 4$ givens

$$\begin{aligned} g(k+1) &= 3g(k) - 4 \\ &= 3(3^k + 2) - 4 && \text{(By the induction hypothesis)} \\ &= 3 \cdot 3^k + 3 \cdot 2 - 4 \\ &= 3^{k+1} + 2 \end{aligned}$$

which completes the induction step and the proof. □