Mathematics 300 Homework, November 15, 2017.

Do problems 25 and 27 on page 170 of the text.

We have proven the following a few weeks ago.

Proposition 1. *Let m be any positive integer and a, b, c, d any integers. If*

$$a \equiv b \mod m \qquad \textit{and} \qquad c \equiv d \mod m$$

then

$$ac \equiv bd \mod m$$

1. Use the Proposition above to prove that for any positive integers m and n and any integers a and b that

$$a^n \equiv b^n \mod m$$

□

I note that we earlier gave solution to the last problem based on the identity $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$. The solution using induction may be a bit more natural for those that do not remember this identity.

2. Let $g(n)$ be defined for all integers $n \geq 0$ and satisfy

$$g(n) = 5g(n-1) - 6g(n-2) \qquad \text{and} \qquad g(0) = -4, \quad g(1) = -7.$$

Prove that

$$g(n) = 3^n - 5(2)^n$$

□

Solution to Problem 1. **Base case:** $n = 1$. Then we are to show $a^1 \equiv b^1 \mod m$. This reduces to $a \equiv b \mod m$, which is what we are given.

Induction hypothesis: $a^k \equiv b^k \mod m$. Then use Proposition 1 with $c = a^k$ and $d = b^k$ to conclude that

$$aa^k \equiv bb^k \mod m$$

Which simplifies to

$$a^{k+1} \equiv b^{k+1} \mod m$$

This is our induction conclusion, which finishes the proof. □

Solution to problem 2. **Base case:** $g(0) = -4$ and $g(1) = -7$.

$$3^0 - 5(2)^0 = 1 - 5 = -4 = g(0)$$
$$3^1 - 5(2)^1 = 3 - 10 = -7 = g(1).$$

So the base case holds.

Induction hypothesis: $g(j) = 3^j - 5(2)^j$ for $1 \leq j \leq k$. Then

$$\begin{aligned} g(k+1) &= 5g(k) - 6g(k-1) \\ &= 5(3^k - 5(2)^k)) - 6(3^{k-1} - 5(2)^{k-1}) \\ &= 5(3^k - 5(2)^k)) - 2 \cdot 3(3^{k-1} - 5(2)^{k-1}) \\ &= 5(3)^k - 25(2)^k - 2(3)^k + 15(2)^k \\ &= 3(3)^k - 10(2)^k \\ &= 3(3)^k - 5 \cdot 2(2)^k \\ &= 3^{k+1} - 5(2)^{k+1}. \end{aligned}$$

which is our induction conclusion. □