Mathematics 300 Homework, November 18, 2017.

Read about strong induction in the text, Section 10.1 pages 161–164. He also does the example we did in class of the number of edges on a tree.

We have already given one proof of

Proposition 1. *If $n \geq 2$ is an integer, then n is divisible by at least one prime number.*

In proving this we will use the fact that if $midk$ and $a \mid n$, then $p \mid k$.

Proof of Proposition 1. We prove this using strong induction.

Base case: $n = 2$. As 2 is prime and it divides itself this case holds.

Induction hypothesis: For every integer k with $2 \leq k < n$ the number is divisible by a prime. Our goal is to show that n is divisible by a prime.

Case 1: n is prime. Then n is prime and $n \mid n$, so n is divisible by a prime.

Case 2: n is not prime. Then $n = k\ell$ where k and ℓ are integers with $2 \leq k, \ell < n$. As $2 \leq k < n$ use the induction hypothesis to conclude thee is a prime p with $p \mid k$. But $k \mid n$. Therefore $p \mid n$. This shows that n is divisible by the prime p and completes the induction. □

We say that n is a ***product of primes*** if $n = p_1 p_2 \cdots p_r$ for some primes $p_1, p_2, \ldots, p_r$. We allow repeats and there may be only one factor in the product. For example:

$$2 = 2, \qquad 6 = 2 \cdot 3, \qquad 60 = 2 \cdot 2 \cdot 3 \cdot 5$$

are all products of primes. In fact we have

Proposition 2. *Every integer $n \geq 2$ is a product of primes.*

Problem 1. Prove this. *Hint:* This is very much like the proof of Proposition 1 above. In Case 2 you can use the induction hypothesis to write each of k and ℓ as a product of primes. □

Problem 2. Do Problem 32 on page 170 of the text. □

Problem 3. Let $y = xe^x$. Then the first several derivatives of y are

$$\begin{aligned} y &= xe^x \\ y' &= (x+1)e^x \\ y'' &= (x+2)e^x \\ y^{(3)} &= (x+3)e^x \end{aligned}$$

Guess a formula for the n-th derivative $y^{(n)}$ and use induction to prove your result. □

Problem 4. This time let $y = xe^{2x}$. As in the last problem compute the first several derivative, guess a formula for $y^{(n)}$ and use induction to prove your result. □