# Mathematics 300 Homework, December 3, 2017.

On pages 200–201 of the text do problems 1, 3,7, and 9.

On page 204 of the text do problems 1,5, 7, and 9.

Recall that if $f\colon A \to B$ is a function, then to show that $f$ is injective we need to show that if $f(a) = f(b)$, then $a = b$. Here to some problems on this.

**1.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 5$. Show that $f$ is injective.
**2.** Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be given by $h(t) = 3 - 2t$. Show that $h$ is injective.

To show that a function $f\colon A \to B$ is not injective you need to find $a \neq b$ with $f(a) = f(b)$.

**3.** Show that the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x, y) = x - y$ is not injective.
**4.** Show that the function $\phi\colon \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = t^2$ is not injective.
**5.** Show that the function $\alpha\colon \mathbb{R} \to \mathbb{R}$ given by $\alpha(x) = e^x(x - 2)(x + 3)$ is not injective.

To show that a function $f\colon A \to B$ is surjective you need to show that for all $b \in B$ that the equation $f(a) = b$ has a solution with $a \in A$. If there is no such solution, then $f$ is not surjective.

**6.** Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = 2t + 1$ is surjective.
**7.** Show that the function $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3 + 1$ is surjective.
**8.** Show that the function $F\colon \mathbb{R} \to \mathbb{R}$ given by $F(r) = r^2 - 2r - 3$ is not surjective.
**9.** Show that the function $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = 2n$ is not surjective.

*Solution to Problem 1:* Let $a, b \in \mathbb{R}$ with $f(a) = f(b)$. Then

$$\begin{aligned} 3a + 5 &= 3b + 5 \\ 3a &= 3b && \text{(subtract 5 from both sides)} \\ a &= b && \text{(divide both sides by 3).} \end{aligned}$$

Thus $f(a) = f(b)$ implies $a = b$ and therefore $f$ is injective. $\square$

*Solution to Problem 2:* Let $a, b \in \mathbb{Z}$ with $h(a) = h(b)$. Then

$$\begin{aligned} 3 - 2a &= 3 - 2b \\ -2a &= -2b && \text{(subtract 3 from both sides)} \\ a &= b && \text{(divide both sides by } -2\text{).} \end{aligned}$$

Therefore $f(a) = f(b)$ implies $a = b$, which shows that $f$ is injective. $\square$

*Solution to Problem 3:* Both $(0, 0)$ and $(1, 1)$ are in the domain of $f$, clearly $(0, 0) \neq (1, 1)$ but

$$f(0, 0) = f(1, 1) = 0.$$

Therefore $f$ is not injective. (There was nothing special about using the points $(0, 0)$ and $(1, 1)$. We could also have used

$$f(3, 7) = f(4, 8) = 4$$

or an infinite number of other possible choices.) $\square$

*Solution to Problem 5:* We have $1 \neq -1$ and

$$\phi(1) = 1^2 = 1 = (-1)^2 = \phi(-1).$$

Therefore $\phi$ is not injective. (Note for any $a \neq 0$ we have $\phi(a) = a^2 = (-a)^2 = \phi(-a)$. So there we choices other than 1 and $-1$.) $\square$

*Solution to Problem 5:* Looking at the function $\alpha(x) = e^x(x - 2)(x + 3$ we see that $\alpha(x) = 0$ has two solution:

$$\begin{aligned} \alpha(2) &= e^2(2 - 2)(2 + 3) = 0 \\ \alpha(-3) &= e^{-3}(2 - (-3))((-3) + 3) = 0. \end{aligned}$$

As $2 \neq -3$ and $\alpha(2) = \alpha(-3) = 0$, we see that $\alpha$ is not injective. $\square$

*Solution to Problem 6:* Let $b \in \mathbb{R}$. To show that $f$ is surjective we need to show that for all $b \in \mathbb{R}$, there is a solution $f(a) = b$ with $a \in \mathbb{R}$. That is we need to solve

$$2a + 1 = b$$

for $a$. This is not hard. I leave the algebra to you:

$$a = \frac{b - 1}{2}.$$

This shows that $f$ is surjective. $\square$

*Solution to Problem 7:* This time we need to show that

$$f(a) = a^3 + 1 = b$$

can be solved for $a$ for any choice of $b$.

$$\begin{aligned} a^3 + 1 &= b \\ a^3 &= b - 1 && \text{(substract 1 from both sides)} \\ a &= \sqrt[3]{b-1} && \text{(take cube roots of both sides).} \end{aligned}$$

Since the cube root of a real number is a real number this shows that for any $b \in \mathbb{R}$ there is a solution of $f(a) = a^3 + 1 = 1$ where $b$ is a real number. Thus $f$ is surjective. $\square$

*Solution to Problem 8:* Our function is $F(r) = r^2 - 2r - 3$. We complete the square on this.

$$F(r) = r^2 - 2r - 3 = r^2 - 2r + 1 - 1 - 3 = (r-1)^2 - 4.$$

As $(r-1)^2 \geq 0$ this implies that

$$F(r) = (r-1)^2 - 4 \geq 0 - 4 = -4.$$

Therefore there is no solution to $F(a) = -5$. Thus $F$ is not surjective. $\square$

*Solution to Problem 9:* Our function is $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2$. Thus if $b \in \mathbb{Z}$ and $f(a) = b$ has a solution we have

$$f(a) = 2a = b.$$

This implies that $b$ is even. Therefore if $b$ is odd, there is no solution to $f(a) = b$ with $a \in \mathbb{Z}$. So $f$ is not surjective.

$\square$