Solutions to two of the October 4 Homework Problems.

This problems have to do with ***congruence modulo*** n. We recall the definition. If a and b are integers and n is a positive integer, then $a \equiv b \pmod{n}$ if $n \mid (b-a)$. Therefore when ask to prove something about congruences, often the first step will be to write out the definition.

1. Let n be a positive integer and a and b any integers. Prove the following:
(a) $a \equiv a \pmod{n}$.
(b) If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.

Solution to (a). Every integer n divides 0. Thus $n \mid (a-a) = 0$. Therefore $a \equiv a \pmod{n}$. □

Solution to (b). We are given that $a \equiv b \pmod{n}$. By definition this means $n \mid (b-a)$. Therefore there is an integer k such that

$$(b-a) = kn$$

Multiply this by -1 to get

$$(a-b) = (-1)(b-a) = (-k)n = \ell n \qquad \text{where } \ell = k \text{ is an integer.}$$

Thus $n \mid (a-b)$ and so $b \equiv a \pmod{n}$ by the definition of congruence. □

2. Let n be a positive integer and a, b, and c any integers. Prove the following:
(a) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.
(b) If $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$.

Solution to (a). We are given that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. Thus, by definition, $n \mid (b-a)$ and $n \mid (c-b)$. Therefore there are integers k and ℓ such that

$$b - a = kn$$
$$c - b = \ell b.$$

Add these two equations to get

$$c - a = (c-b) + (b-a) = kn + \ell n = (k+\ell)n = mn$$

where $m = k + \ell$ is an integer. Therefore $n \mid (c-a)$ and thus $a \equiv c \pmod{n}$. □

Solution to (b). We are given that $a \equiv b \pmod{n}$, which by definition implies that $n \mid (b - a)$. Therefore there is an integer k such that

$$b - a = kn.$$

Then

$$(b + c) - (a + c) = b - a = kn$$

and therefore $n \mid (b+c)-(a+c)$. So by definition $a+c \equiv b+c \pmod{n}$. $\square$