Solution to homework collected on Wednesday, October 11.

Proposition 1. *If r is a rational number and $r \neq 1$. Then*

$$s = \frac{r^3 - 4r + 1}{r - 1}$$

is also a rational number.

Proof. We first write down what we are given. First r is a rational number and so by definition

$$r = \frac{a}{b}$$

where a and b are integers. We are also given $r \neq 1$. This implies $a \neq b$.

We now use this information in our formula for s

$$\begin{aligned}
s &= \frac{r^3 - 4r + 1}{r - 1} \\
&= \frac{\left(\frac{a}{b}\right)^3 - 4\left(\frac{a}{b}\right) + 1}{\left(\frac{a}{b}\right) - 1} \\
&= \frac{\left(\left(\frac{a}{b}\right)^3 - 4\left(\frac{a}{b}\right) + 1\right) b^3}{\left(\left(\frac{a}{b}\right) - 1\right) b^3} && \text{(Mult. top and bottom by } b^3\text{)} \\
&= \frac{a^3 - 4ab^2 + b^3}{(a - b)b^2} \\
&= \frac{p}{q}
\end{aligned}$$

where

$$\begin{aligned}
p &= a^3 - 4ab^2 + b^3 \\
q &= (a - b)b^2
\end{aligned}$$

are integers and $q = (a - b)b^2 \neq 0$ because $a \neq b$ and $b \neq 0$. Therefore s is rational. $\square$