Solutions to homework collected on Friday, October 13.

Problem 1. The number $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$ is irrational.

Solution: In the problem we are assuming that we already know that $\sqrt{2}$ is irrational. Towards a contraction assume that the given number is rational. There there are integers a and b with $b \neq 0$ such that

$$\frac{1+\sqrt{2}}{1-\sqrt{2}} = \frac{a}{b}.$$

We now solve for $\sqrt{2}$ in terms of a and b. First cross multiply to get

$$a(1-\sqrt{2}) = b(1+\sqrt{2}).$$

which is equivalent to

$$(a+b)\sqrt{2} = a - b$$

and thus

$$\sqrt{2} = \frac{a-b}{a+b} = \frac{p}{q}$$

where $p = a - b$ and $q = a + b$ are integers. This implies that $\sqrt{2}$ is rational, contradicting that $\sqrt{2}$ is irrational. $\square$

Problem 2. Let n be an integer with $n \geq 2$. Let d be the smallest integer $d > 1$ which is a factor of n. Show that d is prime.

Solution: Towards a contradiction assume that d is not prime. Then d factors as $d = ab$ where $1 < a, b < d$. We are assuming that d is a divisor of n and therefore $n = qd$ for some integer q. Putting these facts together gives

$$\begin{aligned} n &= qd \\ &= qab \\ &= ka \end{aligned}$$

where $k = qb$ is an integer. Thus a divides n and $1 < a < d$, contradicting that d is the smallest integer > 1 that divides n. $\square$