Solutions to the Mathematics 300 Homework, of October 18.

We have proven the following two theorems.

Theorem 1. *Let n be a positive integer and a and b any integers. If a and b have the same remainder when divided by n, then $a \equiv b \pmod{n}$.*

Theorem 2. *Let n be a positive integer and a and b any integers. If $a \equiv b \pmod{n}$, then a and b have the same remainder when divided by n.*

It is convenient to combine these as one theorem:

Theorem 3. *Let n be a positive integer and a and b any integers. Then $a \equiv b \pmod{n}$ if and only if a and b have the same remainder when divided by n.*

The division algorithm tells us that if a and n are integers with $n > 0$ then there are unique integers q and r such that

$$a = qn + r \qquad \text{where} \qquad 0 \leq r < n.$$

Thus the only possible remainders are when a number is divided by n are $0, 1, \ldots, (n-1)$.

Combining all these facts gives

Theorem 4. *If n is a positive integer, then for any integer a exactly one of the following n cases holds:*

$$\begin{aligned} a &\equiv 0 \pmod{n} \\ a &\equiv 1 \pmod{n} \\ \vdots \ &\ \ \vdots \\ a &\equiv n-1 \pmod{n} \end{aligned}$$

Here is what this means for some small values of n.

Proposition 5. *For any integer a exactly one of the following two cases holds*

$$\begin{aligned} a &\equiv 0 \pmod{2} && \textit{(that is } a \textit{ is odd)} \\ a &\equiv 1 \pmod{2} && \textit{(that is } a \textit{ is even)} \end{aligned}$$

(Whence is a is even if and only if $a \equiv 0 \pmod{2}$ and a is odd if and only if $a \equiv 1 \pmod{2}$.)

Proposition 6. *For any integer a exactly one of the following 3 cases holds*

$$\begin{aligned} a &\equiv 0 \pmod{3} \\ a &\equiv 1 \pmod{3} \\ a &\equiv 2 \pmod{3} \end{aligned}$$

Proposition 7. *For any integer a exactly one of the following 4 cases holds*

$$\begin{aligned} a &\equiv 0 \pmod 4 \\ a &\equiv 1 \pmod 4 \\ a &\equiv 2 \pmod 4 \\ a &\equiv 3 \pmod 4 \end{aligned}$$

and at this point you have gotten idea.

Here are some examples of how these results can be used to simplify some proofs we have done earlier.

Proposition 8. *For any integer n the number $n^3 + 13n$ is even.*

Proof. There are two cased $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod n$.

Case 1. $n \equiv 0 \pmod 2$. Then, using that $13 \equiv 1 \pmod 2$,

$$\begin{aligned} n^3 + 13n &\equiv 0^3 + 13 \cdot 0 && \pmod 2 \\ &\equiv 0 && \pmod 2. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 2$. Then

$$\begin{aligned} n^3 + 13n &\equiv 1^3 + 1 \cdot 1 && \pmod 2 \\ &\equiv 2 && \pmod 2 \\ &\equiv 0 && \pmod 2. \end{aligned}$$

So in all cases $n^3 + 13n \equiv 0 \pmod 2$ and thus $n^3 + 13n$ is always even. □

Problem 1. Use the idea of the last proof to show that for any integer n, the number $n^4 + 11n - 3$ is odd.

Solution. There are two cases: either n is even or n is odd.

Case 1. n is even. Then $n \equiv 0 \pmod 2$. Whence

$$\begin{aligned} n^4 + 11n - 3 &\equiv 0^4 + 11 \cdot 0 - 3 && \pmod 2 \\ &\equiv -3 && \pmod 2 \\ &\equiv 1 && \pmod 2. \end{aligned}$$

Therefore $n^4 + 11n - 3$ is odd.

Case 2. n is odd. Thne $n \equiv 1 \pmod 2$ and thus

$$\begin{aligned} n^4 + 11n - 3 &\equiv 1^4 + 11 \cdot 1 - 3 && \pmod 2 \\ &\equiv 9 && \pmod 2 \\ &\equiv 1 && \pmod 2 \end{aligned}$$

and thus $n^4 + 11n - 3$ is odd in this case also. □

Here is anther one we did earlier:

Proposition 9. *For any integer n we have $3 \,|\, n(n+1)(n+2)$.*

Proof. Showing that $3 \mid n(n+1)(n+2)$ is the same as showing

$$n(n+1)(n+2) \equiv \pmod 3.$$

There are three cases:

Case 1. $n \equiv 0 \pmod 3$. Then

$$\begin{aligned} n(n+1)(n+2) &\equiv (0)(0+1)(0+2) \\ &\equiv 0. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 3$. Then, using that $3 \equiv 0 \pmod 3$, we have

$$\begin{aligned} n(n+1)(n+2) &\equiv (1)(1+1)(1+2) && \pmod 3 \\ &\equiv (1)(2)(3) && \pmod 3 \\ &\equiv (1)(2)(0) && \pmod 3 \\ &\equiv 0 && \pmod 3. \end{aligned}$$

Case 3. $n \equiv 2 \pmod 3$. Then, again using that $3 \equiv 0 \pmod 3$, we have

$$\begin{aligned} n(n+1)(n+2) &\equiv (2)(2+1)(2+2) && \pmod 3 \\ &\equiv (2)(3)(4) && \pmod 3 \\ &\equiv (2)(0)(4) && \pmod 3 \\ &\equiv 0 && \pmod 3. \end{aligned}$$

Thus in all cases $n(n+1)(n+2) \equiv 0 \pmod 3$ and thus $3 \mid n(n+1)(n+2)$ for all integers n. □

Problem 2. Use the idea of the last proof this show that for all integers n that $n(n+1)(n+2)(n+3)$ is dividable by 4.

Solution. Given any integer n exactly one of the congruences

$$\begin{aligned} n &\equiv 0 && \pmod 4 \\ n &\equiv 1 && \pmod 4 \\ n &\equiv 2 && \pmod 4 \\ n &\equiv 3 && \pmod 4 \end{aligned}$$

holds. So we do the proof by consider these four cases.

Case 1. $n \equiv 0 \pmod 4$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &\equiv 0(0+1)(0+2)(0+3) && \pmod 4 \\ &\equiv 0 && \pmod 4 \end{aligned}$$

And this implies $n(n+1)(n+2)(n+3)$ is divisible by 4.

Case 2. $n \equiv 1 \pmod 4$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &\equiv 1(1+1)(1+2)(1+3) && \pmod 4 \\ &\equiv 24 && \pmod 4 \quad \equiv 0 \pmod 4 \end{aligned}$$

Which implies that $n(n+1)(n+2)(n+3)$ is dividable by 4.

Case 3. $n \equiv 2 \pmod 4$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &\equiv 2(2+1)(2+2)(2+3) \pmod 4 \\ &\equiv 144 \pmod 4 \quad \equiv 0 \pmod 4 \end{aligned}$$

Which implies that $n(n+1)(n+2)(n+3)$ is dividable by 4.

Case 4. $n \equiv 3 \pmod 4$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &\equiv 3(3+1)(3+2)(3+3) \pmod 4 \\ &\equiv 360 \pmod 4 \quad \equiv 0 \pmod 4 \end{aligned}$$

Which implies that $n(n+1)(n+2)(n+3)$ is dividable by 4.

Thus is all case $n(n+1)(n+2)(n+3)$ is dividable by 4. □

Problem 3. Show that there are no integers x and y such that

$$x^2 + 3y^4 = 2.$$

Hint: Consider problem $\pmod 3$. Then $3y^4 \equiv 0y^4 \equiv 0 \pmod 3$ holds for all y. So you only need consider the three cases $x \equiv 0 \pmod 3$, $x \equiv 1 \pmod 3$, and $x \equiv 2 \pmod 3$.

Solution. More generally we will show that for any integer a and positive integer b that

$$x^2 + 3ay^b = 2$$

has no solution in integers.

As hinted there are three cases:

Case 1. $x \equiv 0 \pmod 3$. Then, using that $3 \equiv 0 \pmod 3$,

$$\begin{aligned} x^2 + 3ay^b &\equiv 0^2 + 3 \cdot y^b \pmod 3 \\ &\equiv 0 \pmod 3 \\ &\not\equiv 2 \pmod 3 \end{aligned}$$

and therefore $x^2 + 3ay^b = 2$ has no solutions in this case.

Case 2. $x \equiv 1 \pmod 3$. Then, again using that $3 \equiv 0 \pmod 3$,

$$\begin{aligned} x^2 + 3ay^b &\equiv 1^2 + 3 \cdot y^b \pmod 3 \\ &\equiv 1 \pmod 3 \\ &\not\equiv 2 \pmod 3. \end{aligned}$$

Therefore $x^2 + 3ay^b = 2$ has no solutions in this case.

Case 3. $x \equiv 1 \pmod 3$. Then, again using that $3 \equiv 0 \pmod 3$,

$$\begin{aligned} x^2 + 3ay^b &\equiv 2^2 + 3 \cdot y^b \pmod 3 \\ &\equiv 4 \pmod 3 \\ &\equiv 1 \pmod 3 \\ &\not\equiv 2 \pmod 3. \end{aligned}$$

Therefore $x^2 + 3ay^b = 2$ has no solutions in this case either. □

Problem 4. Do Problems 3, 5, and 10 on Page 129 of the text.

Problems 3 and 5 have solutions in the back of the text.

Solution to Problem 10. The problem is to show that if $a \in \mathbb{Z}$, that $a^3 \equiv a \pmod{3}$. I am pretty sure we have done this in class, but here it is again. As you very likely expect we split it into three case, depending on the remainder $\pmod{3}$. That is we consider the three cases $a \equiv 0 \pmod{3}$, $a \equiv 1 \pmod{3}$ and $a \equiv 2 \pmod{3}$.

Case 1. $a \equiv 0 \pmod{3}$. Then

$$\begin{aligned} a^3 &\equiv 0^3 && \pmod{3} \\ &\equiv 0 && \pmod{3} \\ &\equiv a && \pmod{3} \end{aligned}$$

Therefore $a^3 \equiv a \pmod{3}$ holds in this case.

Case 2. $a \equiv 1 \pmod{3}$. Then

$$\begin{aligned} a^3 &\equiv 1^3 && \pmod{3} \\ &\equiv 1 && \pmod{3} \\ &\equiv a && \pmod{3} \end{aligned}$$

And $a^3 \equiv a \pmod{3}$ holds in this case also.

Case 3. $a \equiv 2 \pmod{3}$. Then

$$\begin{aligned} a^3 &\equiv 2^3 && \pmod{3} \\ &\equiv 8 && \pmod{3} \\ &\equiv 2 && \pmod{3} \\ &\equiv a && \pmod{3} \end{aligned}$$

and this $a^3 \equiv a \pmod{3}$ holds in this last case also. □