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Mathematics 300 Test 2 Name:

You are to use your own calculator, no sharing.
Show your work to get credit.

1. (15 Points) (a) Define $a \equiv b \pmod{n}$. (Include in your definition all the conditions that a, b and n must satisfy.)

Let a and b be integers and n be a positive integer. Then, $a \equiv b \pmod{n}$ can be defined as $n \mid (a-b)$.

(b) Use the identity $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ to prove $x \equiv y \pmod{n}$ implies $x^3 \equiv y^3 \pmod{n}$.

Proof: Given that $x \equiv y \pmod{n}$, we can define this expression as: $qn = x - y$ for some integer q. Then we can apply this to acquire,

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$
$$= qn(x^2+xy+y^2)$$
$$= kn$$

since $k = q(x^2+xy+y^2) \in \mathbb{Z}$. Thus, $n \mid (x^3-y^3)$ which tells us that $x^3 \equiv y^3 \pmod{n}$ is in fact true.

Well done! Nice job!

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2. (5 Points) Use that $10 \equiv -1 \pmod{11}$ explain why

$$3{,}642 \equiv -3 + 6 - 4 + 2 \pmod{11}.$$

(The emphasis here is on explaining—so some English will be involved—rather than just giving the answer.)

In this, 3,642 can be expressed as $3(10)^3 + 6(10)^2 + 4(10) + 2$. We express 3,642 in that form to utilize what was given to us in regards to $10 \equiv -1 \pmod{11}$. So from there,

$$3{,}642 \equiv 3(10)^3 + 6(10)^2 + 4(10) + 2 \mod 11$$
$$\equiv 3(-1)^3 + 6(-1)^2 + 4(-1) + 2 \mod 11$$
$$3{,}642 \equiv -3 + 6 - 4 + 2 \mod 11.$$

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3. (8 Points) Let n be an integer. Show that $3n+1$ is odd if and only if n is even.

~~Two cases~~ : ① If $3n+1$ is odd, then, n is even (contrapositive)
Two implications ② If n is even, then $3n+1$ is odd.

case ① (contrapositive) Assume that n is not even, thus it's odd.
This means that $n = 2a+1$ for some integer a.

$$3n+1 = 3(2a+1)+1$$
$$= 6a+3+1 = 6a+4 = 2(3a+2) \approx 2k \text{ where } k = (3a+2).$$

This shows that if n is odd, then $3n+1$ is even, & $\therefore$ not odd.

case ② (direct) Let n be even, where $n = 2a$ for some integer a.

$$3n+1 = 3(2a)+1 = 6a+1 = 2(3a)+1 \approx 2k+1 \text{ where } k = 3a \in \mathbb{Z}$$

This shows that if n is even, then $3n+1$ is odd.

4. (10 Points) Show that for all positive integers $n^4 - n^2$ is divisible by 4. *Hint:* One way to use cases.

$4 \mid (n^4 - n^2)$

$a \equiv 0$
$a \equiv 1$
$a \equiv 2$
$a \equiv 3$

There are 4 cases:

case ① $a \equiv 0 \pmod 4$ $\rightarrow$ $n^4 - n^2 \equiv (0)^4 - (0)^2 \equiv 0 \pmod 4$ ✓

case ② $a \equiv 1 \pmod 4$ $\rightarrow$ $n^4 - n^2 \equiv (1)^4 - (1)^2 \equiv 0 \pmod 4$ ✓

case ③ $a \equiv 2 \pmod 4$ $\rightarrow$
$$n^4 - n^2 \equiv (2)^4 - (2)^2 \pmod 4$$
$$\equiv 16 - 4 \pmod 4$$
$$\equiv 12 \pmod 4$$
$$\equiv 0 \pmod 4$$ ✓

case ④ $a \equiv 3 \pmod 4$ $\rightarrow$
$$n^4 - n^2 \equiv (3)^4 - (3)^2 \pmod 4$$
$$\equiv 81 - 9 \pmod 4$$
$$\equiv 72 \pmod 4$$
$$\equiv 0 \pmod 4$$

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All four cases prove that $n^4 - n^2$ is divisible by 4

5. (12 Points) (a) Define r is a ***rational number***.

A rational number r is defined as

$r = \frac{p}{q}$ where p, q are both integers and $q \neq 0$.

(b) Prove that the sum of two rational numbers is a rational number.

Proof: We want to find the sum of two rational numbers so lets define two rational numbers first,

$r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ where a, b, c, d are all integers and b, d cannot equal zero,

$$\text{Thus, } r_1 + r_2 = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd} = \frac{p}{q}$$

where $p = ad + bc$ and $q = bd$ are both elements of the set $\mathbb{Z}$. In addition, $q \neq 0$. Thus we have shown $r_1 + r_2 = \frac{p}{q}$ implying that $r_1 + r_2$ is rational.

6. (5 Points) Write an English sentence or two explaining why there exist two prime numbers whose sum is 30.

To prove that such a sum exists with two prime numbers, we must provide an example. In this case, two primes such as 23 and 7 have the sum of 30.

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7. (10 Points) The following is true:

Proposition. *If n is an integer and $7 \mid n^2$, then $7 \mid n$.*

Use this to show $\sqrt{7}$ is not a rational number.

Towards a contradiction, assume that $\sqrt{7}$ is rational.

This means that $\sqrt{7} = \frac{a}{b}$, $b \neq 0$, and $\frac{a}{b}$ is in it's lowest terms.

If we square both sides, we get: $7 = \frac{a^2}{b^2}$

$$7b^2 = a^2 \quad (*)$$

This means that $7 \mid a^2$, and by the proposition, we also know that $7 \mid a$. Thus $a = 7x$ for some integer x

use this in $(*)$: $7b^2 = (7x)^2$

$$7b^2 = 49x^2 \quad (\div 7)$$

$$b^2 = 7x^2$$

This shows that $7 \mid b^2$, ~~x~~ $\therefore 7 \mid b$. Thus $b = 7y$ for some integer y

Therefore, $\frac{a}{b} = \frac{7x}{7y}$ for some integers x and y where $y \neq 0$

Since $\frac{a}{b}$ has to be in it's lowest terms, so this is a contradiction.

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8. (15 Points) Let

$$A = \{12x + 18y : x, y \in \mathbb{Z}\}$$
$$B = \{x \in \mathbb{Z} : 6 \mid x\}$$
$$C = \{x \in \mathbb{Z} : 30 \mid x\}$$

(a) Show $C \neq B$.

Assume that $x = 12$, so $6 \mid x$ is true so $x \in B$. But, $30 \nmid x$, so $x \notin C$. Thus, because there is at least one element in B that is not in C, $B \neq C$

(b) Show $A = B$.

We must prove two proofs.

1) $A \subseteq B$

2) $B \subseteq A$

Thus, to prove $\boxed{1}$, assume $p \in A$. Thus, $p = 12x + 18y$, $x, y \in \mathbb{Z}$.
Now, factor out a 6.

$$p = 12x + 18y$$
$$p = 6(2x + 3y)$$
$$p = 6n \text{ for some } n \in \mathbb{Z}.$$

Thus, by definition, $6 \mid p$, which makes $p \in B$, proving that $p \in A$ is $p \in B$, meaning $A \subseteq B$.

Now, to prove $\boxed{2}$, assume $b \in B$. Thus, $6 \mid b$. Now, this means that $b = 6n$ for some $n \in \mathbb{Z}$. Thus,

$$b = 6n$$
$$b = n(12(-1) + 18) \quad \text{because } 12(-1) + 18 = 6$$
$$b = 12(-n) + 18n$$
$$b = 12x + 18y \text{ for } x, y \in \mathbb{Z} \text{ where } x = -n \text{ and } y = n.$$

Thus, $b \in A$, making $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, $B = A$.

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9. (10 Points) (a) For a real number x, define $|x|$.
(b) Prove $|-2x| = 2|x|$.

a) $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

b) Case 1: $x = 0$, then $|x| = 0$ then $|-2x| = |-2(0)| = 0 = 2|0| = 2|x|$.

Case 2: $x > 0$, then $|x| = x$, $|-2x| = 2x = 2|x|$.
as $-2x < 0$

Case 3: $x < 0$, then $|x| = -x$, $|-2x| = |2 \cdot (-x)| = |2x| = 2|x|$.

10. (10 Points) (a) State the ***division algorithm*** and in particular what are the ***quotient*** and ***remainder*** if b is divided into a.

Let a and b be integers with $b > 0$. Then the division algorithm states that there are unique integers q and r [illegible]
such that $a = bq + r$ [illegible]
q = quotient if b is divided into a
r = remainder if b is divided into a

(b) Show that if 42 is divided into n and the remainder is 18 that n is divisible by 6.

$n = 42q + 18$ $\quad n, q \in \mathbb{Z}$

then $n = 6(7q + 3)$

$n = 6k$ where $k = 7q + 3 \in \mathbb{Z}$.

So if 42 is divided into n and the remainder is 18 then n is divisible by 6.

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