90/100

Mathematics 300 Test 3 Name:

Show your work to get full credit.

1. (5 points) Write a sentence or two explaining why 13 is the sum of two perfect squares. (Recall that an integer n is a perfect square if $n = k^2$ for some integer k.)

The integer 13 is a sum of two perfect squares because it is the sum of 4 and 9. The number $4 = 2^2$ and $9 = 3^2$ making 4 & 9 perfect squares.

good

2. (15 points) The rational root test for polynomials of degree four says that if

$$r = \frac{p}{q}$$

is a rational root in lowest terms of the equation

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 = 0$$

where a_0, a_1, a_2, a_2, a_4 are integers then

$$p \mid a_0 \quad \text{and} \quad q \mid a_4.$$

Use this to prove that $\sqrt[4]{2}$ is not a rational number.

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Proof: Lets assume we have the equation $x^4 - 2 = 0$. In this, $a_0 = -2$ and $a_4 = 1$. Thus the only p and q values we have are $p = \pm 1, \pm 2$ $q = \pm 1$. Thus, the only rational roots are $r = \pm 1, \pm 2$. So, we will use these values to see if they are actually roots of $x^4 - 2 = 0$.

$r = -1$: $(-1)^4 - 2 = -1 \neq 0$
$r = 1$: $(1)^4 - 2 = -1 \neq 0$
$r = -2$: $(-2)^4 - 2 = 14 \neq 0$
$r = 2$: $(2)^4 - 2 = 14 \neq 0$

good

So, $x^4 - 2 = 0$ does not have a rational root.
Now, note that $\sqrt[4]{2}$ is a root of $x^4 - 2 = 0$. We just showed that $x^4 - 2 = 0$ does not have a rational root. Thus, $\sqrt[4]{2}$ is not rational; hence, making $\sqrt[4]{2}$ irrational.

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3. (10 points) Prove or give a disproof: There are integers a and b such that

$$10a^3b - 25b = 13.$$

True or false? False

Towards a contradiction assume there are integers a and b such that $10a^3b - 25b = 13$.

Then $5(2a^3b - 5b) = 13$

and $5m = 13$ when $m = 2a^3b - 5b$ for some $m \in \mathbb{Z}$.

This tells us $5 \mid 13$ which is a contradiction, so this statement is false.

good

4. (10 points) Prove or give a disproof: There are integers a and b such that

$$4a + 3b = 1.$$

True or False? True

Let $a = 1$ and $b = -1$

Then $a, b \in \mathbb{Z}$ and $4(1) + 3(-1) = 4 - 3 = 1$

So this statement is true.

5. (10 points) Let R be a relation on the set A. Define the following:

(a) R is ***reflexive***.

This means that xRx for all $x \in A$.

(b) R is ***symmetric***.

This means that if xRy, then yRx for all $x, y \in A$.

(c) R is ***transitive***.

This means that if xRy and yRz, then xRz for all $x, y, z \in A$.

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6. (10 points) On the set of positive integers let R be the relation

$$aRb \iff a \mid b$$

(a) Is R reflective? Prove your answer is correct. Yes or no? Yes.

Yes, R is ~~reflective~~. because for every x, $x|x$.

because $x = 1 \cdot x$
$x = nx$ for $n = 1$

Thus, $x|x$ by definition.

(b) Is R symmetric? Prove your answer is correct. Yes or no? No

No, R is not symmetric. For examble, ~~let~~ a=6 and b=12. Then $a|b$, but $b \nmid a$.

$a|b$ $b|z$ $a|z$

(c) Is R transitive? Prove your answer is correct. Yes or no? Yes

Yes, R is transitive. To prove this, we need to prove if $a|b$ and $b|z$, then $a|z$ because that is the definition of transitive. Then, $b = an$ for some $n \in \mathbb{Z}$ and $z = bm$ for some $m \in \mathbb{Z}$. Now, replace b for an in the last equation. good

$z = bm$
$z = anm$
$z = ap$ for some $p \in \mathbb{Z}$, where $p = \cancel{nm}$. Thus, $a|z$.

7. (5 points) Let a and b be integers such that

$$12a - 5b = 1.$$

Prove that a and b are relatively prime.

If $12a - 5b = 1$, $a, b \in \mathbb{Z}$, then the only integer that can be factored out of ~~both~~ sides is ± 1, because 1's only factors are ± 1. For $12a - 5b = 1$ to be ~~an~~ integer one, you can only factor the same number from ~~each~~ side. Since 1 is the only relative positive factor of 12a and -5b and so is of positive of a and b, a and b are relatively prime.

8. (10 points) Let f be a function defined on the positive integers such that

$$f(n) = 2f(n-1) + 1 \qquad \text{and} \qquad f(1) = 1.$$

Use induction to prove that for all $n \geq 1$

$$f(n) = 2^n - 1.$$

Proof: Base case, $n=1$.

$f(1) = 2^1 - 1 = 2 - 1 = 1 = f(1)$. So the base case holds.

Induction hypothesis: $f(k) = 2^k - 1$, and we want to reach the conclusion of $f(k+1) = 2^{k+1} - 1$.

So, $f(k+1) = 2f(k+1-1)+1$
$= 2f(k)+1$
$= 2(2^k - 1) + 1$
$= 2 \cdot 2^k - 2 + 1$
$f(k+1) = 2^{k+1} - 1$

This concludes our induction.

9. (10 points) Use induction to show that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$. *Hint:* Recall that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Proof: We want to show $3 \mid (n^3 + 2n)$ for all $n \geq 1$.

Base case: $n = 1$. So, $3 \mid (1^3 + 2(1)) = 3 \mid 3$ which is true.

Induction hypothesis: $3 \mid (k^3 + 2k)$ and we want the induction conclusion of $3 \mid (k+1)^3 + 2(k+1)$. So, by definition,

$k^3 + 2k = 3a$ for all $a \in \mathbb{Z}$.

So, $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= k^3 + 2k + 3k^2 + 3k + 3$
$= 3a + 3k^2 + 3k + 3$
$= 3(a + k^2 + k + 1)$
$= 3b$

good

where $b = a + k^2 + k + 1 \in \mathbb{Z}$. This concludes our induction.

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10. (10 points) Use induction to show that if A, B_1, B_2, $\ldots$, B_n are sets then

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

You may use the identity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Base case) Let $n=2$. Then, $A \cap (B_1 \cup B_2)$. Thus, let $B_1 = B$ and $B_2 = C$, making it $A \cap (B \cup C)$. Then, the identity above states $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Thus, replacing original letters, $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$. Base case holds.

Induction Hypothesis) Let $A \cap (B_1 \cup B_2 \cup \ldots \cup B_k) = (A \cap B_1) \cup \ldots \cup (A \cap$

Now, look at $k+1$.

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_k \cup B_{k+1}) = (A \cap (B_1 \cup \ldots \cup B_k)) \cup (A \cap B_{k+1}$$

now, substitute for $(A \cap B_1) \cup \ldots \cup (A \cap B_k)$.

$$= (A \cap B_1) \cup \ldots \cup (A \cap B_k) \cup (A \cap B_{k+1}).$$

Thus, this concludes the induction. good job

11. (5 points) Find the sum of

$$S = b + bs + bs^2 + bs^3 + bs^4 + bs^5 + bs^6.$$

The sum is $\dfrac{b - bs^7}{1-s}$

Multiply S by s. Then,

$$Ss = \cancel{bs} + \cancel{bs^2} + \cancel{bs^3} + \cancel{bs^4} + \cancel{bs^5} + \cancel{bs^6} + bs^7$$

$$- S = b + \cancel{bs} + \cancel{bs^2} + \cancel{bs^3} + \cancel{bs^4} + \cancel{bs^5} + \cancel{bs^6}$$

$$Ss - S = bs^7 - b$$

$$S(s-1) = bs^7 - b$$

$$S = \frac{bs^7 - b}{s-1}$$

$$S = \frac{b - bs^7}{1-s}$$

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