Mathematics 141 Test 2 Name: Key

You are to use your own calculator, no sharing.
Show your work to get credit.

1. (40 Points) Compute the derivatives of the following functions.

(a) $y = \dfrac{3x}{x^2+9}$ $\qquad y' = \dfrac{-3x^2+12}{(x^2+9)^2}$

$$y' = \frac{3(x^2+9) - 3x(2x)}{(x^2+9)^2}$$

$$= \frac{3x^2+12-6x^2}{(x^2+9)^2} = \frac{-3x^2+12}{(x^2+9)^2}$$

(b) $r = 2\sin(3\theta) - 4\cos(5\theta) - 6\tan(7\theta)$ $\qquad \dfrac{dr}{d\theta} = 6\cos(3\theta)+20\sin(5\theta)-42\sec^2(7\theta)$

(c) $f(t) = \sin(t)\cos(t)$ $\qquad f'(t) = \cos^2(t) - \sin^2(t)$

$$f'(t) = \cos(t)\cos(t) + \sin(t)(-\sin(t))$$

$$= \cos^2(t) - \sin^2(t)$$

(d) $y = e^{x^2}\tan x^2$ $\qquad y' = 2xe^{x^2}(\tan^2(x^2) + \sec^2(x^2))$

$$y' = e^{x^2}(2x)\tan(x^2) + e^{x^2}\sec^2(x^2)(2x)$$

$$= 2xe^{x^2}(\tan(x^2) + \sec^2(x^2))$$

(e) $A(r) = r\tan^{-1}(r)$ $\qquad A'(r) = \tan^{-1}(r) + \dfrac{r^2}{1+r^2}$

$$A'(r) = 1\tan^{-1}(r) + r\left(\frac{1}{1+r^2}\right)$$

$$= \tan^{-1}(r) + \frac{r^2}{1+r^2}$$

(f) $y = \cos^{-1}(x)$ $\qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

(g) $h(s) = s\sin^{-1}(s^2)$ $\qquad h'(s) = \sin^{-1}(s^2) + \frac{2s^2}{\sqrt{1-s^4}}$

$h'(s) = (1)\sin^{-1}(s^2) + s\frac{1}{\sqrt{1-(s^2)^2}}(2s)$

(h) $g(x) = x\ln(x) - x$ $\qquad g'(x) = \ln(x)$

$g'(x) = (1)\ln(x) + \frac{x}{x} - 1$

$= \ln(x) + 1 - 1$

$= \ln(x)$

(i) $y = \ln\left(e^{x^2}(x+1)^2\sqrt{x}\right)$ $\qquad \frac{dy}{dx} = 2x + \frac{2}{x+1} + \frac{1}{2x}$

$= x^2 + 2\ln(x+1) + \frac{1}{2}\ln(x)$

$y' = 2x + \frac{2}{x+1} + \frac{1}{2x}$

(j) $w = \ln(\cos z)$ $\qquad \frac{dw}{dz} = -\tan(z)$

$\frac{dw}{dz} = \frac{1}{\cos(z)}(-\sin z)$

$= -\frac{\sin(z)}{\cos(z)}$

$= -\tan(z)$

2. (10 Points) (a) Let x and y be related by $x^2 + 4xy + y^2 = 13$. Find $y' = \frac{dy}{dx}$.

$$\frac{d}{dx}(x^2+4xy+y^2) = \frac{d}{dx} 13$$

$y' =$ $\frac{-x-2y}{2x+y}$

$$2x + 4y + 4xy' + 2yy' = 0$$

$$(4x+2y)y' = -2x-4y$$

$$y' = \frac{-2x-4y}{4x+2y} = \frac{2(-x-2y)}{2(2x+y)} = \frac{-x-2y}{2x+y}$$

(b) What is the equation of the equation of the tangent line to $x^2 + 4xy + y^2 = 13$ at the point $(1, 2)$.

Equ is

The equation is $y-2 = -\frac{5}{4}(x-1)$

$$y - y_0 = m(x-x_0)$$

$$x_0 = 1,\ y_0 = 2$$

$$y = \frac{-x-2y}{2x+y}\Bigg|_{\substack{x=1\\y=2}} = \frac{-1-2(2)}{2(1)+2} = -\frac{5}{4}$$

3. (10 Points) (a) What is the linearization approximation of $f(x) = \sqrt{8+x}$ at the point $x = 1$?

The linear approximation is

$f(x) \approx$ $3 + \frac{1}{6}(x-1)$

$$f(x) \approx f(a) + f'(a)(x-a)$$

Here $a = 1$

$$f(1) = \sqrt{8+1} = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2\sqrt{8+x}}$$

$$f'(1) = \frac{1}{2\sqrt{8+1}} = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

$$f(x) \approx 3 + \frac{1}{6}(x-1)$$

(b) Use this approximation to estimate $f(1.3)$. (Show your work.)

By (a)

$f(1.3) \approx$ 3.05

$$f(1.3) \approx 3 + \frac{1}{6}(1.3-1)$$

$$= 3 + \frac{1}{6}(.3)$$

$$= 3 + .05$$

$$= 3.05$$

4. (5 Points) Find all solutions to $y' = 3x^2 + 2x$.

$y =$ x^3+x^2+C

One solution is

$y = x3+x2$

to get all solutions add

a constant, C

5. (10 Points) The surface area of a cube is increasing at a constant rate of 2 in^2/hour. Let s be the side length of the cube.

(a) At what rate is the side length of the cube increasing?

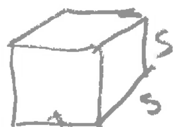
$\frac{ds}{dt} =$ $\frac{1}{6s}$ in/hr

Let s = side length

in inches

A = surface area in^2

V = volume in^3



on side

has area s^2

6 sides

so $A = 6s^2$

$A = 6s^2$

$V = s^3$

Given $\frac{dA}{dt} = 2$

But $\frac{dA}{dt} = \frac{d}{dt} 6s^2 = 12s\frac{ds}{dt} = 2$

so $\frac{ds}{dt} = \frac{2}{12s} = \frac{1}{6s}$

(b) At what rate is the volume changing when the side is 5 inches long?

The rate is: 2.5 in^3/hr

$V = s^3$

$\frac{dV}{dt} = \frac{d}{dt} s^3 = 3s^2\frac{ds}{dt}$

$= 3s^2\left(\frac{1}{6s}\right)$

$= \frac{s}{2}$

So when $s = 5$

$\frac{dV}{dt} = \frac{5}{2} = 2.5$

6. (10 Points) (a) State the Mean Value Theorem:

If f is continuous on $[a,b]$ and differentiable on (a,b) then there is a $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

(b) Use the Mean Value Theorem to show that if f is differentiable on and interval and $f' > 0$ on the interval, then f is increasing on the interval. (That is if $x_2 > x_1$, then $f(x_2) > f(x_1)$.)

Let $x_2 > x_1$. By mean value theorem (with $a = x_1$, $b = x_2$) we have

$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} = f'(c) > 0 \quad \text{for some } c$$

Multiply by the positive number $x_2 - x_1$ to get

$$f(x_2) - f(x_1) > 0$$

i.e. $f(x_2) > f(x_1)$

7. (10 Points) Let $f(x)$ satisfy

$$f'(x) = e^x(x-1)(x-2)(4-x).$$

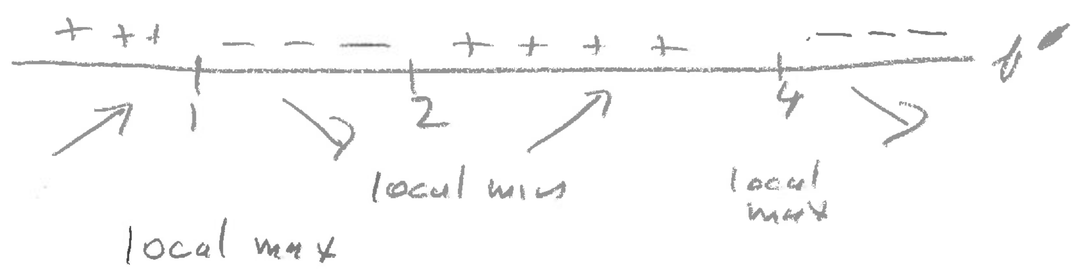
$f'(x) = e^x(x-1)(x-2)(4-x)$ gives us critical points $x = 1, 2, 4.$

What are the critical points of f? 1, 2, 4

What are the local maximizers of f? 1, 4

What are the local minimizers of f? 2

(These are solutions to $f'(x) = 0$)



8. (10 Points) Find the absolute maximum and minimum of $f(x) = xe^{-x/2}$ with $0 \leq x \leq 5$. ***Show all your work to get credit for this problem.***

Maximum is $\frac{2}{e}$

Maximizer(s) is 2

Minimum is 0

Minimizer(s) is 0

First find the critical points

$$f'(x) = (1)e^{-\frac{x}{2}} + x\, e^{-\frac{x}{2}}\left(-\tfrac{1}{2}\right)$$

$$= e^{-\frac{x}{2}}\left(1 - \frac{x}{2}\right) = 0$$

so $1 - \frac{x}{2} = 0$

$x = 2$

$$f(2) = 2e^{-2/2} = 2e^{-1} = \frac{2}{e}$$

$+ + +$ | $- - - -$ f'

$\nearrow$ 2 $\searrow$

so $x = 2$ is maximizer.

At end points

$$f(0) = 0e^{-0/2} = 0$$

$$f(5) = 5e^{-\frac{5}{2}} > 0$$

so $f(0) = 0$ is minimum