NOTES ON ANALYSIS

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Contents

1. Metric Spaces.

Definition 1. A ***metric space*** is a nonempty set E with a function $d\colon E\times E\to[0,\infty)$ such that for all $p,q,r\in E$ the following hold

(a) $d(p,q)\geq 0$,
(b) $d(p,q)=0$ if and only if $p=q$,
(c) $d(p,q)=d(q,p)$, and
(d) $d(p,r)\leq d(p,q)+d(q,r)$. □

The function d is called the ***distance function*** on E. The condition $d(p,q)=d(q,p)$ is that the distance between points is ***symmetric***. The inequality $d(p,r)\leq d(p,q)+d(q,r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E\subseteq\mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p,q)=|p-q|.$$

Problem 1. Show that this makes E into a metric space. □

We have seen that if $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ are points in $\mathbb{R}^n$ and we define the ***length*** or ***norm*** of p to be

$$\|p\|=\sqrt{p_1^2+\cdots+p_n^2}$$

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then the inequality

$$\|p+q\| \leq \|p\| + \|q\|$$

holds.

Proposition 2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p, q \in E$ let*

$$d(p,q) = \|p-q\|.$$

Then E with the distance function d is a metric space.

Problem 2. Prove this. □

Here are some inequalities that we will be using later.

Proposition 3 (Reverse triangle inequality)**.** *Let E be a metric space with distance function d and let $x, y, z \in E$. Then*

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

Problem 3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. □

Proposition 4. *Let E be a metric space with distance function d and $x_1, \ldots, x_n \in E$. Then*

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Problem 4. Prove this. Then draw a picture in the plane showing why this is reasonable. *Hint:* Induction. □

Definition 5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

(a) The ***open ball*** of radius r centered at x is

$$B(a,r) := \{x : d(a,x) < r\}.$$

(b) The ***closed ball*** or radius r centered at a is

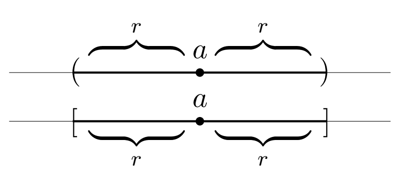
$$\overline{B}(a,r) := \{x : d(a,x) \leq r\}.$$

□

In the real numbers with their usual metric $d(x,y) = |x-y|$ the open and closed balls about a are intervals with center a:

$$B(a,r) = (a-r, a+r) =$$

$$\overline{B}(a,r) = [a-r, a+r] =$$



In the plane, $\mathbb{R}^2$, with its usual metric $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$, the open and closed balls about $\mathbf{a}$ are disks centered at $\mathbf{a}$.

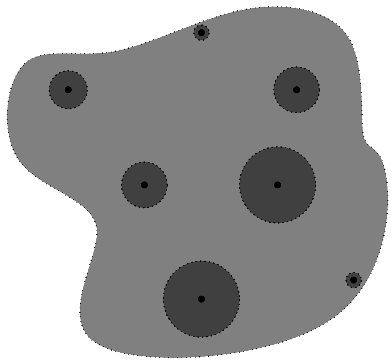
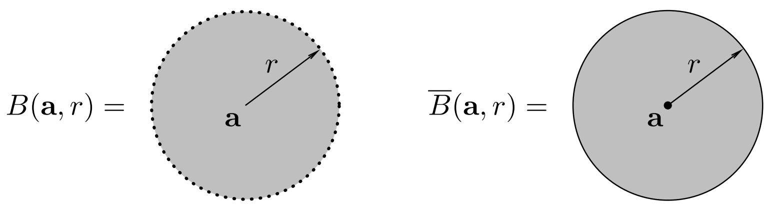


FIGURE 1. A set is open if and only if each of its points is the center of an open ball contained in the set.



Definition 6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. □

This can be restated by saying that S is open if and only if each of its points is the center of an open ball contained in S. See Figure 1.

Proposition 7. *In any metric space E, the sets E and $\varnothing$ are open.* □

Proof. Let $p \in E$, then for any $r > 0$ we have $B(p, r) = \{x \in E : d(x, p) < r\} \subseteq E$. Thus E contains not only some open ball about p, it contains every open ball about p. Therefore E is open.

That $\varnothing$ is open is a case of a vicious implication. To see this consider the statement

$$p \in \varnothing \text{ and } r > 0 \implies B(p, r) \subseteq \varnothing.$$

If this statement is true, then $\varnothing$ satisfies the definition of beiing open. But this is a true statement as an implication $P \implies Q$ is true whenever the hypotheses, P, is false. And the hypothesis "$p \in \varnothing$ and $r > 0$" is false as "$p \in \varnothing$" is false. □

Proposition 8. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x, r)$ is an open set.*

Problem 5. Prove this. *Hint:* Let $x \in B(a, r)$. Then $d(a, x) < r$. Set $\rho := r - d(a, x) > 0$ and show $B(x, \rho) \subseteq B(a, r)$ □

Proposition 9. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a, b) are open.*

Problem 6. Prove this. □

Proposition 10. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a,r))$, of the closed ball $\overline{B}(a,r)$ is open.*

Proposition 11. *Prove this.* Hint: *If $x \in \mathcal{C}(B(a,r))$, then $d(x,a) > r$. Let $\rho := d(a,x) - r > 0$ and show $B(a,\rho) \subseteq \mathcal{C}(B(a,r))$.* □

Proposition 12. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is an $r > 0$ such $B(x,r) \subseteq U$. But then $B(x,r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x,r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x,r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then

$$B(x,r) \subseteq B(x,r_1) \subseteq U \quad \text{and} \quad B(x,r) \subseteq B(x,r_2) \subseteq V$$

and therefore $B(x,r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. □

Proposition 13. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i \in I} U_i$ is open.

(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 7. Prove this. □

Problem 8. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

Definition 14. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. □

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 15. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. □

Proposition 16. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a,r)$ is closed.* □

Problem 9. Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. □

Proposition 17. *If E is a metric space, then every finite subset of E is closed.*

Problem 10. Prove this. $\square$

Problem 11. In the real numbers show that the half open interval $[0, 1)$ is neither open or closed. $\square$

Problem 12. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m, n) = |m - n|$. Note that for this metric space if $m \neq n$ that $d(m, n)$ is a nonzero positive integer and thus $d(m, n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n, r)$ is the one element set $B(n, r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n \in S}\{n\}$ and use Proposition 13 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. $\square$

Proposition 18. *Let E be a metric space.*

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i \in I} F_i$ is closed.
(b) Let $F_1, \dots, F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 13. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 13. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1) \cap \mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1) \cap \mathcal{C}(F_2))$$

is closed. $\square$

Let E be a metric space. Then a function $f\colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(p) - f(q)| \leq M d(p, q) \quad \text{for all } p, q \in E.$$

Proposition 19. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for all $c \in \mathbb{R}$ the sets*

$$f^{-1}\big[(c, \infty)\big] = \{p \in E : f(p) < c\}$$
$$f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) > c\}$$

are open and the sets

$$f^{-1}\big[[c, \infty)\big] = \{p \in E : f(p) \geq c\}$$
$$f^{-1}\big[(-\infty, c]\big] = \{p \in E : f(p) \leq c\}$$

are closed.

Half of the proof. Assume that f satisfies $|f(p)-f(q)| \leq Md(p,q)$ for $p, q \in E$. We will show that $f^{-1}\big[(-\infty, c)\big]$ is open. We need to show that for any $q \in f^{-1}\big[(-\infty, c)\big]$ the set $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about q. As $q \in f^{-1}\big[(-\infty, c)\big]$ we have $f(q) < c$. Therefore

$$r = \frac{c - f(q)}{M}$$

is positive. Let $pe \in B(q, r)$. Then

$$\begin{aligned}
f(p) &= f(q) + (f(p) - f(q)) \\
&\leq f(q) + |f(p) - f(q)| && (\text{as } (f(p) - f(q)) \leq |f(p) - f(q)|) \\
&\leq f(q) + Md(p,q) && (\text{as } f \text{ is Lipschitz}) \\
&< f(q) + Mr && (\text{as } p \in B(q,r), \text{ so } d(p,q) < r) \\
&= f(q) + M\left(\frac{c - f(q)}{M}\right) && (\text{from our definition of } r) \\
&= c.
\end{aligned}$$

Therefore if $p \in B(q,r)$ we have $f(p) < c$ and thus $B(q,r) \subseteq f^{-1}\big[(-\infty, c)\big]$. Whence $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about any of its points q and therefore $f^{-1}\big[(-\infty, c)\big]$ is open.

We now show $f^{-1}\big[[c, \infty]\big] = \{p \in E : f(p) \geq c\}$ is closed. We know $f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) < c\}$ is open. Its compliment is

$$\mathcal{C}\left(f^{-1}\big[(-\infty, c)\big]\right) = f^{-1}\big[[c, \infty)\big].$$

Therefore $f^{-1}\big[[c, \infty)\big]$ is the compliment of an open set, which means that $f^{-1}\big[[c, \infty)\big]$ is closed. □

Problem 14. Prove the other half of Proposition 19, that is show $f^{-1}\big[(c, \infty)\big]$ is open and $f^{-1}\big[(-\infty, c]\big]$ is closed. □

Proposition 20. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set*

$$f^{-1}[c] = \{p \in E : f(p) = c\}$$

is a closed set.

Problem 15. Prove this. *Hint:* Write $f^{-1}[c]$ as the intersection of two closed sets. □

We can now give some more examples of open and closed sets in the plane, $\mathbb{R}^2$ where we are using the distance function $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$. Let $\mathbf{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$ define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + b$$

where b is a real number. If $\mathbf{x} = (x_1, x_2)$ this can be written in vector form as

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ then

$$\begin{aligned} |f(\mathbf{p}) - f(\mathbf{q})| &= |\mathbf{a} \cdot \mathbf{p} + b - (\mathbf{a} \cdot \mathbf{q} + b)| \\ &= |\mathbf{a} \cdot (\mathbf{p} - \mathbf{q})| \\ &\leq \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\| && \text{(Cauchy-Schwartz)} \\ &= M d(\mathbf{p}, \mathbf{q}) \end{aligned}$$

where $M = \|\mathbf{a}\|$. Thus f is a Lipschitz function.

Consider the case where $\mathbf{a} = (1, 0)$ and $b = 0$. Then for any $c \in \mathbb{R}$. In this case $f(x_1, x_2) = x_1$, or in slightly different notation $f(x, y) = x$. Therefore Proposition 19 implies the sets

$$\{(x, y) : x > c\}, \quad \{(x, y) : x < c\}$$

are open and that

$$\{(x, y) : x \geq c\}, \quad \{(x, y) : x \leq c\}$$

are closed.

Problem 16. Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 0\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \geq 0, x + y \leq 0\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. $\square$

1.1. Definition of limit in a metric space and some special limits in $\mathbb{R}$.

Definition 21. Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots \rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that

$$n > N \quad \Longrightarrow \quad d(p_n, p) < \varepsilon.$$

In the case we say that the sequence $\langle p_n \rangle_{n=1}^{\infty}$ ***converges*** to p. □

Problem 17. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. □

Here are some examples of working with limits in $\mathbb{R}$.

Example 22. If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} 5x_n = 5x$.

Proof. Let $\varepsilon > 0$. Note that

$$|5x_n - 5x| = 5|x_n - x|.$$

From the definition of $\lim_{n\to\infty} x_n = x$ there is a $N > 0$ such that

$$n > N \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{5}.$$

But then (multiply by 5)

$$n > N \quad \text{implies} \quad |5x_n - 5x| < \varepsilon.$$

But this is just the definition of $\lim_{n\to\infty} 5x_n = 5x$. □

Proposition 23. *Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$

Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y.$$

Proof. Let $\varepsilon > 0$. Then from the definition of $\lim_{n\to\infty} x_n = x$, there is a $N_1 > 0$ such that

$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{2}.$$

Likewise $\lim_{n\to\infty} y_n = y$ implies there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2}.$$

Set

$$N = \max\{N_1, N_2\}.$$

If $n > N$, then $n > N_1$ and $n > N_2$ and thus

$$|(x + y) - (x_n + y_n)| \le |x - x_n| + |y - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is

$$n > N \quad \text{implies} \quad |(x+y) - (x_n + y_n)| < \varepsilon$$

which is exactly the definition of $\lim_{n\to\infty}(x_n + y_n) = x + y$. □

Proposition 24. *Let $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ be sequences in $\mathbb{R}$ with*

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Then for any real numbers a and b

$$\lim_{n\to\infty}(ax_n + by_n) = ax + by.$$

Problem 18. Prove this. □

Proposition 25. *If $\langle x_n\rangle$ be a convergent sequence in $\mathbb{R}$. Then $\langle x_n\rangle$ is bounded. That is a constant M such that $|x_n| \le M$ for all M.* □

Problem 19. Prove this. *Hint:* Let $\varepsilon = 1$. Then there is a N such that

$$n > N \quad \text{implies} \quad |x - x_n| < 1.$$

Therefore is $n > N$ we have

$$|x_n| = |x + (x_n - x)| \le |x| + |x - x_n| < |x| + 1.$$

This bounds all the terms with $n > N$. Let

$$M = \max\big\{|x| + 1, |x_1|, |x_2|, \dots, |x_N|\big\}.$$

Then $|x_n| \le M$ for all n, which shows that the sequence is bounded. □

Theorem 26. *Let*

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n y_n = xy.$$

Problem 20. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - xy_n + xy_n - xy| && \text{(Adding and subtracting trick.)}\\
&\le |x_n y_n - xy_n| + |xy_n - xy|\\
&= |x_n - x||y_n| + |x||y_n - y|
\end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n\rangle_{n=1}^\infty$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad \text{for all} \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. □

Corollary 27. *If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then*

$$\lim_{n\to\infty} x_n^2 = x^2.$$

Proof. Use $\langle x_n \rangle = \langle y_n \rangle$ in Theorem 26. □

Proposition 28. *Let k be a positive integer and $\langle p_n \rangle$ a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} p_n = p$. Then*

$$\lim_{n\to\infty} p_n^k = p^k$$

Problem 21. Prove this. *Hint:* What is probably the easiest why is to use induction. □

Problem 22. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the quadratic polynomial $f(x) = ax^2 + bx + c$ where a, b, c are constants. Let $\langle p_n \rangle$ be a convergent sequence, $\lim_{n\to\infty} p_n = p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p).$$

□

Lemma 29. *Let $a \in \mathbb{R}$ with $a \neq 0$. Let $|x| > \dfrac{|a|}{2}$. Then*

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2},$$

$$\frac{1}{|x|} < \frac{2}{|a|},$$

and

$$\left| \frac{1}{x} - \frac{1}{a} \right| \leq \frac{2|x-a|}{|a|^2}.$$

Problem 23. Prove this. □

Proposition 30. *Let $\langle x_n \rangle$ be a sequence with $\lim_{n\to\infty} = a$ and $a \neq 0$. Then*

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}.$$

Problem 24. Prove this. *Hint:* First note there is a N_1 such that

$$n > N_1 \quad \text{implies} \quad |x_n - a| < \frac{|a|}{2}.$$

Now let $\varepsilon > 0$ There is also a N_2 such that

$$n > N_2 \quad \text{implies} \quad |x_n - a| < \frac{|a|^2}{2}\varepsilon.$$

Now let $N = \max\{N_1, N_2\}$ and use the last lemma to show that

$$n > N \quad \text{implies} \quad \left|\frac{1}{x_n} - \frac{1}{a}\right| < \varepsilon.$$

□

Proposition 31. *Let E be a metric space and $f\colon E \to \mathbb{R}$ be a Lipschitz map. (That is there is a constant M such that for all $p, q \in E$ the inequality $|f(p) - f(q)| \leq M d(p,q)$ holds.) Let $\langle p_n \rangle$ be a sequence in E with $\lim_{n\to\infty} p_n = p$ where $p \in E$. Then*

$$\lim_{n\to\infty} f(p_n) = f(p).$$

Problem 25. Prove this. □

1.1.1. *Limits and rational functions.* We show that limits play will with polynomials and rational functions. Recall a polynomial, $f(x)$, is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers. If $a_n \neq 0$, then the ***degree*** of $f(x)$ is $\deg f(x) = n$. The following is trivial, but useful in doing induction proofs involving polynomials.

Proposition 32. *Let $f(x)$ be a polynomial of degree $n \geq 1$. There there is a polynomial $g(x)$ of degree $(n-1)$ and a constant c such that*

$$f(x) = xg(x) + c.$$

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ Then factor out x from all the non-constant terms:

$$\begin{aligned} f(x) &= x\big(a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1\big) + a_0 \\ &= xg(x) + c \end{aligned}$$

where $g(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1$ and $c = a_0$. □

Theorem 33. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$, say*

$$\lim_{k\to\infty} p_k = p.$$

Then for any polynomial $f(x)$

$$\lim_{k\to\infty} f(p_k) = f(p).$$

Problem 26. Prove this. *Hint:* Use induction on $\deg f(x)$. The base case is $\deg f(x) = 0$, that is $f(x) = a_0$ is a constant. This case $f(p_k) = a_0$ for all n and thus $\langle f(p_k)\rangle_{k=1}^{\infty}$ is a sequence of constants and so the result it true in this case. The case of $\deg f(x) = 1$ is also easy. In this case $f(x) = a_1x + a_0$. And from some some of our earlier results we have

$$\begin{aligned}\lim_{k\to\infty} f(p_k) &= \lim_{k\to\infty}(a_1p_k + a_0)\\ &= a_1p + a_0\\ &= f(p).\end{aligned}$$

Now do the induction step. Assume we know the result is true for polynomials of degree $(n-1)$ and let $f(x)$ be a polynomial of degree n. By Proposition 32 write

$$f(x) = xg(x) + c$$

where $g(x)$ is a polynomial of degree $(n-1)$ and c is a constant. Then by the induction hypothesis we have

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Now use our earlier results about limits of products and sums to finish the induction step and complete the proof. $\square$

Lemma 34. *Let $g(x)$ be a polynomial and $p \in \mathbb{R}$ a point with $g(p) \neq 0$. Let $\langle p_k\rangle_{k=1}^{\infty}$ a sequence with*

$$\lim_{k\to\infty} p_k = p.$$

Then $g(p_k) \neq 0$ for all but at most finitely many k's, and this the sequence

$$\left\langle \frac{1}{g(p_k)} \right\rangle_{k=1}^{\infty}$$

is defined for all but finitely many values of k and

$$\lim_{k\to\infty} \frac{1}{g(p_k)} = \frac{1}{g(p)}.$$

Problem 27. Prove this. *Hint:* First show that $g(p_k) \neq 0$ for all but finitely may k. One way to do this is to let $\varepsilon = |g(p)|/2$ in the definition of a limit. We know from Theorem 33 that

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Let $\varepsilon = |g(p)|/2$ in the definition of $\lim_{k\to\infty} g(p_k) = g(p)$. to find a $N > 0$ such that

$$n > N \quad \text{implies} \quad |g(p_k) - g(p)| < |g(p)|/2.$$

Use this to show

$$n > N \quad \text{implies} \quad |g(k)| > |g(p)|/2$$

and therefore

$$n > N \quad \text{implies} \quad g(p_k) \neq 0.$$

You should now be able to use Proposition 30 to finish the proof.

A ***rational function*** is a function

$$h(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials, $g(x)$ is not identically zero and the domain of $h(x)$ is the set of points where $g(x) \neq 0$.

Theorem 35. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$,*

$$\lim_{k\to\infty} p_k = p.$$

Let

$$h(x) = \frac{f(x)}{g(x)}$$

be a rational function with $g(p) \neq 0$. Then

$$\lim_{k\to\infty} h(p_k) = h(p).$$

Problem 28*.* Prove this by putting together Lemma 34 and Proposition 30. □

We can now do some limits you recall from calculus. For example let us compute

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6}.$$

Divide the numerator and denominator of the fraction in the limit to get

$$\frac{3n^2 - 3n + 7}{4n^2 + 6} = \frac{3 - 3(1/n) + 7(1/n)^2}{4 + 6/(1/n)^2} = \frac{f(1/n)}{g(1/n)}$$

where $f(x)$ and $g(x)$ are the polynomials

$$f(x) = 3 - 3x + 7x^2 \quad g(x) = 4 + 6x^2.$$

And have seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore Theorem 35 gives

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6} = \lim_{n\to\infty} \frac{f(1/n)}{g(1/n)} = \frac{f(0)}{g(0)} = \frac{3}{4}.$$

Problem 29*.* Find the following limits and give a justification (which can just be quoting the right proposition or theorem) for your answer.

(a) $\displaystyle \lim_{n\to\infty} \frac{4n^3 + 5n_6}{7n^3 - 8n + 7}$

(b) $\displaystyle \lim_{n\to\infty} \frac{-3n^2 + 1}{7n^5 - 19}$

□

Problem 30. Let $0 < a < 1$. Give a ε, N proof that

$$\lim_{n\to\infty} a^k = 0.$$

Hint: Let $\varepsilon > 0$. We have proven that for $a \in (0,1)$ there is a natural number N with $a^N < \varepsilon$. □

Problem 31. Let $a > 0$ and $x \geq a/4$. Show

$$|\sqrt{x} - \sqrt{a}| \leq \frac{2|x-a|}{3\sqrt{a}}.$$

□

Proposition 36. *If $\langle a_n \rangle_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$, say*

$$\lim_{n\to\infty} a_n = a$$

with $a > 0$. Then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

Problem 32. Give a N, ε proof of this. □

1.2. **Using limits to show sets are closed.** The next few definitions, propositions, and problems are practice in using the definitions.

Definition 37. Let E be a metric space and S a subset of E. Then $p \in E$ is an ***adherent point*** of s if and only if every open ball about p contains at least one points of S.

Problem 33. To get a feel for what this means, do the following

(a) Show that every point of S is an adherent point of S.
(b) What are the adherent points of the open interval $(0,1)$?
(c) What are the adherent points of the rational numbers, $\mathbb{Q}$, in the real numbers $\mathbb{R}$? □

Theorem 38. *A set is closed if and only if it contains all its adherent points.*

Problem 34. Prove this. *Hint:* Let S be a subset of the metric space E.

(a) First show that if S is closed that it contains all its adherent points. So assume S is closed and p is an adherent point of S. Towards a contradiction assume that p is not in F. Then $p \in \mathcal{C}(S)$, the compliment of S. As S is closed, the set $\mathcal{C}(S)$ is open. Therefore there is an $r > 0$ such that $B(a,r) \subseteq \mathcal{C}(S)$. Show that this contradicts that p is an adherent point of S.
(b) Now show that if S is not closed, then S has an adherent point, p, with $p \notin S$. For if S is not closed, $\mathcal{C}(S)$ is not open and therefore $\mathcal{C}(S)$ has a point $p \in \mathcal{C}(S)$ such that no open ball about p is is contained in $\mathcal{C}(S)$. Show that p is an adherent point of S. □

In what follows we will often want to show that some set is closed. The following gives method for going this that works well in 87.3% of known proofs.

Theorem 39. *Let S be a subset of the metric space E. Then the following are equivalent.*

(a) S is closed.
(b) S contains the limits of its sequences in the sense that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converges, say $x = \lim_{n\to\infty}$, then $x \in S$.

Remark 40. In practice it is the implication (b) $\implies$ (a) that is useful is showing that sets are closed. □

Lemma 41. *Let S be a set in a metric space and p an adherent point of S. Then there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ from S that converges to p.*

Problem 35. Prove this. *Hint:* Let p be an adherent point of S. This means that for every $r > 0$ the ball $B(p, r)$ contains a point of S. For each positive integer n let $p_n \in S$ be point of S that is in the ball $B(p, 1/n)$. Now show that $\lim_{n\to\infty} p_n = p$. □

The converse of the last lemma is also true.

Lemma 42. *Let S be a set in a metric space and p a point that is a limit of a sequence of points from S. Then p is an adherent point of S.*

Problem 36. Prove this. *Hint:* Let S be a set in the metric space E and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S that converges to the point $p \in E$. You need to show that p is an adherent point of S. That is if $r > 0$ the ball $B(a, r)$ contains a point of S. As $\lim_{n\to\infty} p_n = p$ we can use $\varepsilon = r$ in the definition of limit to see there is a $N > 0$ such that $n > N$ implies that $d(p_n, p) < r$. □

Problem 37. Prove Theorem 39. *Hint:*

(a) $\implies$ (b). Assume that S is closed and that $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converge to the point p. Use some of the lemmas above to show that p is an adherent point of S and then use that closed sets contain their adherent points. □

(b) $\implies$ (a). Assume that (b) holds. We wish to show that S is closed. It is enough to show that S contains all its adherent points. Let p be an adherent point of S. Then use one or more of the lemmas above to show that p is a limit of a sequence form S. □

Problem 38. This is an example of using Theorem 39 to to a set is closed. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a polynomial. Then we have seen (Theorem 33) that if $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} f(p_n) = f(p)$. Let F be a closed subset of $\mathbb{R}$ and f a polynomial. Show that

$$S := f^{-1}[F] = \{x : f(x) \in F\}$$

is a closed subset of $\mathbb{R}$. *Hint:* Use Theorem 39. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S with $\lim_{n\to\infty} p_n = p$. To show that S is closed we need to show that $p \in S$. All we need to to is show that $p \in S$. By the definition of S we have $f(p_n) \in F$. Also we have $f(p) = \lim_{n\to\infty} f(p_n)$. Use Theorem

39 to show that $f(p) \in F$, and therefore $p \in f^{-1}[F]$. Now use Theorem 39 again to conclude that $S = f^{-1}[F]$ is closed. □

1.3. **Cauchy sequences, definition of completeness of metric spaces.** The following is one of the basic ideas in analysis.

Definition 43. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then the sequence is a ***Cauchy sequence*** if and only if for all $\varepsilon > 0$, there is a $N > 0$ such that $m, n > N$ implies $d(p_m, p_n) < \varepsilon$. □

A brief version would be that $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall \varepsilon > 0 \ \exists N > 0 \big[\, m, n > N \quad \Longrightarrow \quad d(p_m, p_n) < \varepsilon \,\big].$$

Proposition 44. *Every convergent sequence is a Cauchy sequence.*

Problem 39. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space and let p be its limit. Let N be so that

$$n > N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

Then show that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

The converse is not true. There are Cauchy sequences that are not convergent.

Problem 40. Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n \rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of E. □

You may feel that the example of the last problem is a bit of a cheat as the sequence does converge in the larger space of all real numbers. And is some sense this is true, given a metric space, E, there is a natural way to expand it to a somewhat larger space that contains the limits of all Cauchy sequences from E.

Proposition 45. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Problem 41. Prove this. *Hint:* Let $\varepsilon > 0$. As the sequence is Cauchy, there is a N such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \frac{\varepsilon}{2}.$$

Let $n > N$, then for any k we have by the triangle inequality that

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p).$$

Now show that it is possible to choose k such that both $d(p_n, p_{n_k})$ and $d(p_{n_k}, p)$ are less than $\varepsilon/2$. □

Proposition 46. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space E. Let $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence of this sequence. Then $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is also convergent and has the same limit at the original sequence.*

Problem 42. Prove this. *Hint:* For all k we have $n_k \geq k$. □

Definition 47. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Then this sequence is ***monotone increasing*** if and only if $x_n \leq x_{n+1}$ for all n. It is ***monotone decreasing*** if and only if $x_n \geq x_{n+1}$ for all n. It is ***monotone*** if it is either monotone increasing or monotone decreasing. □

Theorem 48. *A bounded monotone sequence in $\mathbb{R}$ is convergent.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a bounded monotone sequence. We first assume that it is monotone increasing. Let

$$S = \{x_n : n = 1, 2, \ldots\}$$

be the set of values of the sequence. As the sequence is bounded, this set is bounded. Therefore, by Least Upper bound Axiom, this set has a least upper bound $b = \sup(S)$. We now show that the sequence converges to b.

Let $\varepsilon > 0$. Then $b - \varepsilon < b$ and b is the least upper bound of S, therefore $b - \varepsilon$ is not an upper bound for S. Whence there is positive integer N such that $b - \varepsilon < x_N$. Then for any $n > N$ we have

$$\begin{aligned} b - \varepsilon &< x_N \\ &\leq x_n \qquad (x_N \leq x_n \text{ as the sequence is monotone increasing.}) \\ &\leq b \qquad (\text{as } b \text{ is an upper bound for } S \text{ and } x_n \in S.) \end{aligned}$$

Therefore we have $b - \varepsilon < x_n \leq b$ for all $n > N$. Thus $n > N$ implies $|x_n - b| < \varepsilon$ and thus $\lim_{n\to\infty} x_n = b$. □ □

Problem 43. Modify the last proof so show that if $\langle x_n \rangle_{n=1}^{\infty}$ is bounded and monotone decreasing that it converges to $\inf\{x_n : n = 1, 2, 3, \ldots\}$. □

Theorem 49. *Every sequence of real numbers has a monotone subsequence.*

Problem 44. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Call x_n a ***peak point*** if $x_n \geq x_m$ for all $m > n$. (That is x_n is greater than or equal to all the values that follow it.)

Case 1: *There are infinitely many peak points.* In this case there is an infinite subsequence of the sequence consisting of peak points. Show this subsequence is monotone decreasing.

Case 2: *There are only finitely many peak points.* Let N be the largest n such that x_n is a peak point. Thus if $n > N$ the point x_n is not a peak point and therefore there is $m > n$ with $x_n > x_n$. Let $n_1 = N_1$. Then $n_1 > N$ and so there is a $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. But then $n_2 > N$ and thus there is $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. Continue in this manner to show that there is an infinite increasing subsequence. □

Proposition 50. *Let E be a metric space. Then every Cauchy sequence in E is bounded. (That is the sequence is contained in some ball.)*

Problem 45. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\varepsilon = 1$ (or any other positive number that you like). Then there is $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon = 1.$$

Let $a = x_{N+1}$ and set

$$r = 1 + \max\{1, d(a, x_1), d(a, x_2), \dots, d(a, x_N)\}.$$

Then show that $p_n \in B(a, r)$ for all n. $\square$

Theorem 51. *Every Cauchy sequence in* $\mathbb{R}$ *converges.*

Problem 46. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then by Proposition 50 this sequence is bounded. By Theorem 49 this sequence has a monotone subsequence. By Theorem 48 this monotone subsequence converges. Put these facts together with Proposition 45 to prove that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ converges. $\square$

This property of a metric space, that Cauchy implies convergent, is important enough to give a name.

Definition 52. The metric space E is ***complete*** if and only if every Cauchy sequence in E converges. $\square$

So we can restate Theorem 51 as

Proposition 53. *The real numbers,* $\mathbb{R}$*, with their usual metric is a complete metric space.* $\square$

1.4. **Completeness of $\mathbb{R}^n$.** We can get more examples by looking at closed subsets of complete metric spaces.

Proposition 54. *Let E be a metric space and F a closed subset of E. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of points of F that converges in E to some point p. Then $p \in F$. (A nice restatement of this is that a closed set contains all its limit points.)*

Problem 47. Prove this. *Hint:* Towards a contradiction assume that $p \notin F$. Then as F is closed, the compliment $\mathcal{C}(F)$ is open. As $p \in \mathcal{C}F$ by the definition an open set, there is a $r > 0$ such that $B(p, r) \subseteq \mathcal{C}(F)$. But $\lim_{n\to\infty} p_n = p$ and therefore if we let $\varepsilon = r$ there is a $N > 0$ such that $n > 0$ implies $d(p_n, p) < \varepsilon = r$. This this leads to a contradiction. $\square$

Proposition 55. *Let E be a complete metric space and F a closed subset of E. Then F, considered as a metric space in its own right, is complete.*

Problem 48. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence from F. As E is complete this sequence converges to some point, p, of E. To finish the proof it is enough to show that $p \in F$. $\square$

Recall that we have made $\mathbb{R}^n$ into metric spaces with the metric

$$d(p,q) := \|p-q\|$$

where

$$\|p\| = \|(p_1, p_2, \dots, p_n)\| = \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}.$$

Theorem 56. *With this metric $\mathbb{R}^n$ is complete.*

Problem 49. Prove this in the case of $n = 3$. *Hint:* Here is the proof for $n = 2$. Let $\langle p_n \rangle_{n=1}^\infty = \langle (x_n, y_n) \rangle_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^2$. Note we have the inequality

$$|x_m - x_n| = \sqrt{(x_n - x_n)^2} \leq \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} = d(p_m, p_n)$$

with a similar calculation showing

$$|y_m - y_n| \leq d(p_m, p_n).$$

As $\langle p_n \rangle_{n=1}^\infty = \langle (x_n, y_n) \rangle_{n=1}^\infty$ is Cauchy there is a $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

From the inequalities above this gives

$$m, n > N \quad \text{implies} \quad |x_m - x_n|, |y_m - y_n| \leq \|p_m - p_n\| < \varepsilon.$$

Therefore both of the sequences $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ are Cauchy and as $\mathbb{R}$ is complete this implies that they both converge. Thus there are $x, y \in \mathbb{R}^n$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

and thus there are $N_1 > 0$ and $N_2 > 0$ such that

$$\begin{aligned} n > N_1 \quad &\text{implies} \quad |x_n - x| < \frac{\varepsilon}{\sqrt{2}} \\ n > N_2 \quad &\text{implies} \quad |y_n - y| < \frac{\varepsilon}{\sqrt{2}} \end{aligned}$$

Then if $N = \max\{N_1, N_2\}$ and $p = (x, y)$

$$\begin{aligned} n > N \quad \text{implies} \quad \|p_n - p\| &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &< \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} \\ &= \varepsilon. \end{aligned}$$

which shows that the Cauchy sequence $\langle p_n \rangle_{n=1}^\infty = \langle (x_n, y_n) \rangle_{n=1}^\infty$ has the limit (x, y). As this was an arbitrary Cauchy in $\mathbb{R}^2$, this shows $\mathbb{R}^2$ is complete.

Now you do the proof for $\mathbb{R}^3$. $\square$

1.5. **Sequential compactness and the Bolzano-Weierstrass Theorem.**

Definition 57. A subset S of a metric space is ***sequentially compact*** if and only if every sequence $\langle p_n \rangle_{n=1}^{\infty}$ of points from S has a subsequence that converges to a point of S. □

Problem 50. Let S be a finite subset of a metric space. Then S is sequentially compact. *Hint:* Let $S = \{s_1, s_2, \dots, s_m\}$. For each $j \in \{1, 2, \dots, m\}$ let $\mathcal{N}_j = \{n : p_n = s_j\}$. As the union of the sets $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m$ it the infinite set $\mathbb{N} = \{1, 2, 3, \dots\}$ at least one of them is infinite. Say that $\mathcal{N}_j$ is infinite, $\mathcal{N}_j = \{n_1, n_2, n_3, \dots\}$ and consider the subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$. □

Theorem 58 (BolzanoWeierstrass Theorem)**.** *Every closed bounded subset of* $\mathbb{R}$ *is sequentially compact.*

Problem 51. Prove this. *Hint:* Let S be a closed bounded subset of $\mathbb{R}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of points from S. Then the sequence is bounded (because S is bounded). Also (Theorem 49) this sequence has a monotone subsequence. At some point in finishing the proof you will need to use Proposition 54. □

Corollary 59 (BolzanoWeierstrass Theorem for sequences)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be bounded sequence in $\mathbb{R}$. As the sequence is bounded there is a closed ball $\overline{B}(a, r) = [a - r, a + r]$ that contains $\langle x_n \rangle_{n=1}^{\infty}$. The set $\overline{B}(a, r)$ is a closed bounded subset of $\mathbb{R}$ and so by the Bolzano-Weierstrass the $\langle x_n \rangle_{n=1}^{\infty} \langle x_n \rangle_{n=1}^{\infty}$ has a convergent subsequence. □

Theorem 60 (General BolzanoWeierstrass Theorem)**.** *Every closed bounded subset of* $\mathbb{R}^n$ *is sequentially compact.*

Lemma 61. *Let* $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n) \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}^2$*. Then this sequence converges if and only if both the sequences*

$$\langle x_n \rangle_{n=1}^{\infty} \quad \textit{and} \quad \langle y_n \rangle_{n=1}^{\infty}$$

converge.

Proof. Let $p = (x, y) \in \mathbb{R}^2$. Then, as we saw in Problem 49 the inequalities

$$|x_n - x|, \ |y_n - y| \le \sqrt{(x - x_n)^2 + (y_n - y)^2} = d(p_n, p)$$

Therefore if $\langle p_n \rangle_{n=1}^{\infty}$ converges, as $\lim_{n \to \infty} p_n = p$ then for any $\varepsilon > 0$ there is a $N > 0$ such that $n > N$ implies $d(p_n, p) < \varepsilon$. Therefore for this N we have

$$n > N \quad \text{implies} \quad |x_n - x|, |y_n - y| \le d(p_n, p) < \varepsilon$$

and therefore we have $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$.

Conversely assume that both the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, say $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Therefore there are N_1 and N_2 such that

$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and}$$

Thus is $N = \max\{N_1, N_2\}$ we have, just as in Problem 49,

$$n > N \quad \text{implies} \quad d(p_n, p) < \varepsilon$$

which shows that $\langle p_n \rangle_{n=1}^{\infty}$ converges. Now show the limit is in S. $\square$

Lemma 62. *Let $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n, z_n) \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^3$. Then this sequence converges if and only if all three of they sequences*

$$\lim_{n\to\infty} x_n, \quad \lim_{n\to\infty} y_n, \quad \text{and} \quad \lim_{n\to\infty} z_n$$

converge.

Problem 52. Prove Theorem 60 for $n = 3$. *Hint:* Here is the proof for $n = 2$. Let S be a closed bouneded subset of $\mathbb{R}^n$ and $\langle p_n \rangle_{n=1}^{\infty} = \langle (x_n, y_n \rangle_{n=1}^{\infty}$ a sequence in S. As S is bounded the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is bounded. But

$$|x_n|, \; |y_n| \le \sqrt{|x_n|^2 + |y_n|^2} = d(\vec{0}, p_n)$$

and therefore both of the sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ are bounded. As the sequence $\langle x_n \rangle_{n=1}^{\infty}$ is bounded by Corollary 59 it has a convergent subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$. The sequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence of the bounded sequence $\langle y_n \rangle_{n=1}^{\infty}$ and therefore $\langle y_{n_k} \rangle_{k=1}^{\infty}$ is also bounded. Therefore we can use Corollary 59 again to get a subsequence $\langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$ of the subsequence $\langle y_{n_k} \rangle_{k=1}^{\infty}$ $\langle y_{n_k} \rangle_{k=1}^{\infty}$ such that $\langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$ converges. Now note that the subsequence $\langle x_{n_{k_j}} \rangle_{j=1}^{\infty}$ is a convergent subsequence of the convergent sequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$. But a subsequence of a convergent subsequence is convergent (Proposition 46) and therefore $lax_{n_{k_j}} \rangle_{j=1}^{\infty}$ is convergent. But then both of the sequences

$$\langle x_{n_{k_j}} \rangle_{j=1}^{\infty} \quad \text{and} \quad \langle y_{n_{k_j}} \rangle_{j=1}^{\infty}$$

converge and therefore by Lemma 61 this implies the sequence

$$\langle p_{n_{k_j}} \rangle_{j=1}^{\infty} = \langle (x_{n_{k_j}}, y_{n_{k_j}}) \rangle_{j=1}^{\infty}$$

converges. Let $p = \lim_{n\to\infty} p_{n_{k_j}}$. Then, as S is closed, Proposition 54 implies $p \in S$. As $\langle p_n \rangle_{n=1}^{\infty}$ was any sequence from the closed bounded set, S, this shows that every sequence from a closed bounded subset of $\mathbb{R}^2$ has a subsequence that converges to a point of S. Therefore closed bounded subsets of $\mathbb{R}^n$ are sequentially compact. $\square$

Corollary 63 (General Bolzano-Weierstrass for sequences)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Problem 53. Prove this. *Hint:* See the proof of Corollary 59. $\square$

Proposition 64. *Every sequentially compact subset of a metric space is closed and bounded.*

Problem 54. Prove this. *Hint:* Let S be sequentially compact in E. First show that S is bounded. Towards a contradiction assume that it is not bounded. Let q be any point of S. Because S is not bounded there for each positive integer n there is a point $p_n \in S$ with $d(q, p_n) > n$. The set S is sequentially compact and therefore the sequence $\langle p_n \rangle_{n=1}^{\infty}$ has a convergent subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$. Let $p = \lim_{k\to\infty} p_{n_k}$. Then using $\varepsilon = 1$ in the definition of limit we have that there is a $K > 0$ such that $k > K$ implies that $d(p, p_{n_k}) < 1$. Whence for all $k > K$ we have by the triangle inequality

$$n_k < d(q, p_{n_k}) \leq d(q, p) + d(p, p_{n_k}) < d(q, p) + 1$$

which gives a contradiction (why?)

Now use sequential compactness to show S is closed. One way is to show that sequential compactness implies that S contains all its adherent points. $\square$

Remark 65. We have seen, Theorem 60, that every closed bounded of $\mathbb{R}^n$ is sequentially compact. And the last proposition shows that a sequentially compact subset is closed and bounded. But it is important to realize that not all closed bounded subsets of all subsets of all metric spaces are sequentially compact. The next problem give an example. $\square$

Problem 55. Let $E = (0, \infty)$ and let $S = (0, 1]$. Here we are using the metric $d(x, y) = |x - y|$. Show that S is a closed bounded subset of E, but that S is not sequentially compact. $\square$

1.6. **Open covers and the Lebesgue covering lemma.** We recall a bit of set theory. Let E a set and $\mathcal{U}$ a collection of subsets of E. (At bit more formally if $U \in \mathcal{U}$ then $U \subseteq E$.) The ***union*** of $\mathcal{U}$ is

$$\bigcup \mathcal{U} = \{x : x \in U \text{ for at least one } U \in \mathcal{U}\}.$$

We will sometimes use the notation

$$\bigcup_{U \in \mathcal{U}} U$$

or some trivial variants of this notation. For example

$$\bigcup_{n=1}^{\infty} (-n, n) = (-\infty, \infty)$$

or

$$\bigcup_{x \in [0,1]} (x - 1, x + 1) = (-1, 2).$$

Of course there is the ***intersection*** of $\mathcal{U}$ which it

$$\bigcap \mathcal{U} = \{x : x \in U \text{ for all } U \in \mathcal{U}\}.$$

which can also be written as

$$\bigcap_{U \in \mathcal{U}} U$$

with such variants as

$$\bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n) = [a, b]$$

(which gives anther example of an infinite intersection of open sets not being open), and

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \varnothing.$$

Definition 66. Let E be a metric space and $S \subseteq E$. Then $\mathcal{U}$ is an ***open cover*** of S if and only if the following hold

(a) Each element, U, of $\mathcal{U}$ is an open subset of E.
(b) $S \subseteq \bigcup \mathcal{U}$. □

Anther way to say $S \subseteq \cup\mathcal{U}$ is that for all $x \in S$ there is an $U \in \mathcal{U}$ with $x \in U$. This is nothing more than a restatement of the definition of the union, but in practice is how we often work with open covers.

Theorem 67 (Lebesgue Covering Theorem)**.** *Let S be a sequentially compact subset of the metric space E and let $\mathcal{U}$ be an open over of S. Then there is a $r > 0$ (often called a **Lebesgue number** of the cover) such that for all $x \in S$ there are is a $U \in \mathcal{U}$ with $B(x, r) \subseteq U$.*

A restatement is that given an open cover $\mathcal{U}$ of a sequentially compact set S there is a $r > 0$ (which depends on both S and $\mathcal{U}$) such that every point of S is contained in a ball of radius r that is contained in some open set $U \in \mathcal{U}$. In practice this means that in working with open covers, we can sometimes replace them with a cover by balls all with the same radius.

Problem 56. Prove Theorem 67. *Hint:* Towards a contradiction assume that there is an open cover $\mathcal{U}$ of a sequentially compact set S where the Lebesgue Covering Theorem does not hold. This means that for all $r > 0$ there is a point $x \in S$ such that the ball $B(x, r)$ is not contained in any $U \in \mathcal{U}$.

For each positive integer n let $x_n \in S$ be a point where the ball $B(x_n, 1/n)$ is not contained in any of the sets $U \in \mathcal{U}$. As S is sequentially compact, the sequence $\langle x_n \rangle_{n=1}^{\infty}$ has a convergent subsequence, $\langle x_{n_k} \rangle_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = x$ where $x \in S$. As $x \in S$ and $\mathcal{U}$ is an open cover of S there is some $U \in \mathcal{U}$ with $x \in U$. As U is open there is a $r > 0$ such that $B(x, r) \subseteq U$. Because $\lim_{k\to\infty} x_{n_k} = x$ there is a $N > 0$ such that

$$k > N \quad \text{implies} \quad d(x_{n_k}, x) < \frac{r}{2}.$$

Now show that if we choose k such that both $k > N$ and $1/n_k < r/2$ hold then

$$B(x_{n_k}, 1/n_k) \subseteq B(x, r) \subseteq U$$

and explain why this leads to a contradiction. $\square$

1.7. **Open covers and compactness.**

Definition 68. Let S be a subset of the metric space E. Then S is ***compact*** if and only if every open cover of S has a finite subcover. Explicitly the means that if $\mathcal{U}$ is an open cover of S then there is a finite set $\{U_1, U_2, \dots, U_m\} \subseteq \mathcal{U}$ with

$$S \subseteq U_1 \cup U_2 \cup \cdots \cup U_m$$

$\square$

Theorem 69. *Every sequentially compact set in a metric space is compact.*

Problem 57. Prove this. *Hint:* Towards a contradiction assume that S is a sequentially compact subset of some the metric space E that is not compact. That is there is some open cover of $\mathcal{U}$ of S that as no finite subcover. Let r be a Lebesgue number for this open cover. That is for every $p \in S$ there is some $U \in \mathcal{U}$ such that $B(p, r) \subseteq U$. We know that such an r exists by the Lebesgue Covering Theorem 67. Define a sequence of points $p_1, p_2, p_3, \dots \in S$ and a sequence $U_1, U_2, U_3, \dots \in \mathcal{U}$ follows. Let p_1 be any element of S. Then there is a $U_1 \in \mathcal{U}$ such that $B(p_1, r) \subseteq U_1$. Now assume that $p_1, 2_2, \dots, p_n \in S$ and $U_1, U_2, \dots, U_n \in \mathcal{U}$ have been defined such that

$$p_j \notin U_1 \cup U_2 \cup \cdots \cup U_{j-1} \quad \text{and} \quad B(x_j, r) \subseteq U_j$$

for $j = 1, 2, \dots, n$. Now

(a) Explain why there is an $p_{n+1} \in S$ such that $p_{n+1} \notin U_1 \cup U_1 \cup \cdots U_n$.
(b) There is a $U_{n+1} \in \mathcal{U}$ such that $B(p_{n+1}, r) \subseteq U_{n+1}$.

Finish the proof by showing that if $m \neq n$, say $m < n$, then $p_n \notin U_m$ and $B(p_m, r) \subseteq U_m$ implies that $d(p_m, p_n) \geq r$ and thus the sequence $\langle p_n \rangle_{n=1}^{\infty}$ has no convergent subsequence (if $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence has $d(p_{x_k}, p_{x_\ell}) \geq r$ for $k \neq \ell$. Use this to show the subsequence is not Cauchy) which contradicts that S is sequentially compact. $\square$

The converse of the last Theorem is also true.

Theorem 70. *Every compact set in a metric space is sequentially compact.*

Lemma 71. *Let S be a compact set in a metric space and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then there is a point $p \in S$ such that for all $r > 0$ the set*

$$\{n : p_n \in B(p, r)\}$$

is infinite.

Problem 58. Prove this. *Hint:* This is a very typical use of compactness in a proof. Towards a contradiction assume that there is a compact set S

where this does not hold. Then for each $x \in S$ there is a $r_x > 0$ such that $\{n : p_n \in B(x, r_x)\}$ is finite. Set

$$\mathcal{U} = \{B(x, r_x) : x \in S\}.$$

Show that $\mathcal{U}$ is an open cover of S. By compactness there is a finite subcover, say that

$$S \subseteq B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \cdots \cup B(x_m, r_{x_m}).$$

Now for each natural number $n \in \mathbb{N}$ we have that $p_n \in B(x_j, r_{x_j})$ for at least one $j \in \{1, 2, \ldots, m\}$. Thus, by the pigeon hole principle, there is at least one j where the set $\{n : p_n \in B(x_j, r_{x_j})\}$ is infinite. Explain why this is a contradiction. $\square$

Problem 59. Prove this. *Hint:* Let S be a compact set in a metric space E. We wish to show that every sequence $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to a point of S. Towards this end let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. By Lemma 71 there is a point $p \in S$ such that for all $r > 0$ the set of $n \in \mathbb{N}$ with $p_n \in B(x, r)$ is infinite. Show this implies $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to p. $\square$

1.8. **Connected sets.** We can to describe sets that are connected in the everyday sense that they do not split into smaller pieces. Here is the precise definition.

Definition 72. The metric space is ***connected*** if and only if E is not the disjoint union of two nonempty open sets. $\square$

To be very explicit this means there are no open sets A and B with

$$\begin{aligned} E &= A \cup B \\ A \cap B &= \varnothing \\ A &\neq \varnothing \\ B &\neq \varnothing \end{aligned}$$

If a metric space is not connected it is ***disconnected***. Let E be disconnected, then there are open sets A and B, both nonempty with

$$E = A \cup B \quad \text{and} \quad A \cap B = \varnothing. \tag{1}$$

It is convenient to have a name for such a splitting of disconnected space. If E is disconnected, then a ***disconnection*** of E is a pair of nonempty open sets A and B such that (1) holds.

Proposition 73. *If $E = A \cup B$ is a disconnection of the metric space E, then the sets A and B are both open and closed.*

Problem 60. Prove this. *Hint:* Note that $E = A \cap B$ and $A \cap B = \varnothing$ together imply that $\mathcal{C}(A) = B$ (the complement of A in E is B) and $\mathcal{C}(B) = A$. The sets A and B are open by the definition of disconnection. But then $A = (B)$ and $B = (A)$ and the compliment of an open set is closed. $\square$

Call a set in a metric space ***clopen*** if and only if it is both open and closed.

Proposition 74. *Let E be a metric space. Then the following are equivalent.*

(a) E is connected.
(b) There is no decomposition $E = A \cup B$ with A and B both closed, $A \neq \varnothing$, $B \neq \varnothing$ and $A \cap B = \varnothing$.
(c) The only clopen sets in E are E and $\varnothing$.

Problem 61. Prove this. □

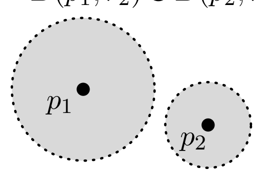
Let us look at some examples. First we will look at examples of disconnected sets.

Problem 62. Let X be a metric space, $p_1, p_2 \in X$ and $r_1, r_2 > 0$ with

$$r_1 + r_2 \leq d(p_1, p_2), Let$$

and set

$$E = B(p_1, r_2) \cup B(p_2, r_2).$$



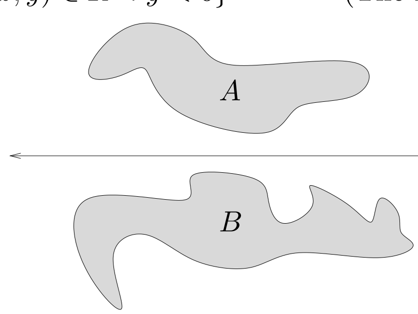
Show E is disconnected. *Hint:* As $r_1 + r_2 \leq d(p_1, p_2)$ the two balls are disjoint. And they are both open. □

Proposition 75. *Let $E \subseteq \mathbb{R}$. Say that E is split if there is a point $p \notin E$ such that there are $a, b \in E$ with $a < p < b$. Then that if E is split, it E is disconnected.*

Proof. Prove this. *Hint:* Let $A = (-\infty, p) \cap E$ and $B = E \cap (p, \infty)$. Then $a \in A$ and $b \in B$ so A and B are nonempty, Show that A and B are open subsets of E. □ □

Problem 63. Here is a two dimensional analogue of the last proposition. Let

$$U = \{(x, y) \in \mathbb{R}^2 : y > 0\} \qquad \text{(The upper half plane)}$$
$$V = \{(x, y) \in \mathbb{R}^2 : y < 0\} \qquad \text{(The lower half plane).}$$



Let A be a nonempty subset of U and B a nonempty subset of V and let $E = A \cup B$. Show that $E = A \cup B$ is a disconnection of E and therefore E is disconnected. $\square$

Lemma 76. *Let E be a metric space and $S \subseteq E$ a subset of E. Then S is metric space in its on right (using the distance function from E). Let $A \subseteq S$ be a subset of S. Then A is open in S if and only if there is an open set U in E such that $A = U \cap S$.*

Proof. For a point $a \in S$ let

$$\begin{aligned} B_E(a,r) &= \{x \in E : d(x,a) < r\} \\ B_S(a,r) &= \{x \in S : d(x,a) < r\} \end{aligned}$$

be the open balls of radius r about a in the spaces E and S respectively. Then

$$B_S(a,r) = S \cap B_E(a,r).$$

First assume that $A = S \cap U$ where U is open in E and let $a \in A$. Then, by the definition of U being open in E, there is a $r > 0$ such that $B_E(a,r) \subseteq U$. But then

$$B_S(a,r) = S \cap B_E(a,r) \subseteq S \cap U = A.$$

Thus A contains the ball $B_S(a,r)$. As a was any point of A this shows that S contains a ball of S about any of its points and therefore A is open in S.

Conversely assume that A is open in S. We wish to find an open set U of E such that $A = S \cap U$. By the definition of A being open in S for each $a \in A$ there is a $r_a > 0$ such that

$$B_S(a, a_r) \subseteq A.$$

This implies

$$\begin{aligned} A &= \bigcup_{a \in A} B_S(a, r_r) \\ &= \bigcup_{a \in A} (S \cap B_E(a, r_a)) \\ &= S \cap \bigcup_{a \in A} B_E(a, r_a) \\ &= S \cap U \end{aligned}$$

where

$$U = \bigcup_{a \in A} B_E(a, r_a)$$

is a union of open balls of E and therefore is an open set in E. $\square$

Proposition 77. *Let E be a metric space and for $\alpha \in I$ let S_α be a connected subset of E where I is some index set. Assume that for all $\alpha, \beta \in I$ that*

$$S_\alpha \cap S_\beta \neq \varnothing.$$

Then the union

$$S = \bigcup_{\alpha \in I} S_\alpha$$

is connected.

Proof. Towards a contradiction assume that S is not connected, and let

$$S = A \cup B$$

be a disconnection of S where A and B are open in S, each is nonempty and $A \cap B = \varnothing$. As A and B are nonempty there is are $a \in A$ and $b \in B$. As $S = \bigcup_{\alpha \in I} S_\alpha$, there are $\alpha, \beta \in I$ with $a \in S_\alpha$ and $b \in S_\beta$. Then $a \in A \cap S_\alpha$ and therefore $A \cap S_\alpha \neq \varnothing$. We now claim that $S_\alpha \cap B = \varnothing$. For if $S_\alpha \cap B \neq \varnothing$ then, as $S_\alpha \subseteq S = A \cup B$, implies

$$S_\alpha = (S_\alpha \cap A) \cup (S_\alpha \cap B). \tag{2}$$

The sets A and B are open in S, and therefore, by Lemma 76, the sets $A \cap S_\alpha$ and $S_\alpha \cap B$ are open in S_α. This implies that (2) is a disconnection of S_α, contradicting that S_α is connected. But $S_\alpha \subseteq A \cup B$ and $S_\alpha \cap B = \varnothing$ implies $S_\alpha \subseteq A$.

A similar argument shows that $S_\beta \subseteq B$. Whence

$$S_\alpha \cap S_\beta \subseteq A \cap B = \varnothing$$

contradicting that $S_\alpha \cap S_\beta \neq \varnothing$. □

We have yet to get a nontrivial example of a connected space.

Theorem 78. *Let I be an interval in $\mathbb{R}$. Then I is connected.*

Proof. We will use the following property of the interval I: if $a, b \in I$ with $a < b$, then $[a, b] \subseteq I$.

Now assume, towards a contradiction, that I is disconnected and let

$$I = A \cup B$$

be a disconnection of I. As A and B are nonempty there are elements $a \in A$ and $b \in B$. By relabeling if need be we can assume that $a < b$. Then $[a, b] \subseteq I$ and therefore $I = A \cup B$ implies

$$[a, b] = ([a, b] \cap A) \cup ([a, b] \cap B)$$

is a disconnection of $[a, b]$. To simplify notation we replace I by $[a, b]$ by $[a, b] \cap A$ by A and $[a, b] \cap B$ by B and assume that we have a disconnection

$$[a, b] = A \cup B \quad \text{with} \quad a \in A \quad \text{and} \quad b \in B.$$

By Proposition 73 the sets A and B are both closed in $[a, b]$.

The set A is a subset of the interval $[a, b]$ and therefore A is bounded. So by the least upper bound axiom A has a least upper bound.

$$\alpha = \sup(A).$$

As we are in $\mathbb{R}$ the ball $B(\alpha, r)$ is just the interval $(\alpha - r, \alpha + r)$. If For any $r > 0$ we must have $B(\alpha, r) \cap A \neq \varnothing$, for otherwise $a - r$ would be an upper bound for A, contradicting that α is the least upper bound. Therefore α is an adherent point of A. As A is closed this implies that $\alpha \in A$.

Also for all $r > 0$ we have $B(\alpha, \beta) \cap B \neq \varnothing$. For if $(\alpha - r, \alpha + r) \cap B = \varnothing$, then $(\alpha - r, \alpha + r) \cap [a, b] \subseteq A$, which implies that A contains points x with $x > \alpha$, contradicting that α is an upper bound for A. Thus α is an adherent point of B and B is closed. Thus $\alpha \in B$.

But then we have $\alpha \in A \cap B$, which contradicts $A \cap B = \varnothing$. □

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