Mathematics 554H/701I Homework

Contents

1. Continuous functions.

1.1. **Continuity of a function at a point and some examples.** We now start the last big topic we will cover this term, which is continuous maps between metric spaces.

Definition 1. Let E and E' be metric spaces and $f\colon E \to E'$ a function from E to E'. Let $p_0 \in E$. Then f is ***continuous*** at p_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $p \in E$

$$d(p, p_0) < \delta \quad \text{implies} \quad d(f(p), f(p_0)) < \varepsilon.$$

□

Example 2. Here is an example of showing something is continuous. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = 3x + 5$$

Then f is continuous at every point of $\mathbb{R}$. To see this let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta = \varepsilon/3$. Then if $|x - x_0| < \delta$ we have

$$\begin{aligned} |f(x) - f(x_0)| &= |3x + 5 - (3x_0 + 5)| \\ &= |3(x - x_0)| \\ &= 3|x - x_0| \\ &< 3\delta \\ &= \varepsilon. \end{aligned}$$

Proposition 3. *Let E be a metric space and $f\colon E \to E$ the identity map, that is $f(p) = p$ for all $p \in E$. Then f is continuous at all points of E.*

Problem 1. Prove this. □

Problem 2. Let E be a metric space.

(a) Let $p, x_0, q \in E$ show that

$$|d(q, x_0) - d(p, x_0)| \leq d(p, q).$$

(b) Let $x_0 \in E$ and define $f(p)$ to be the distance of p from x_0, that is $f(p) = d(p, x_0)$. Show that f is continuous at all points of E. *Hint:* Use part (a) to show $|f(p) - f(q)| \le d(p, q)$. $\square$

Recall that a map $f \colon E \to E'$ between metric spaces is ***Lipschitz*** if and only if there is a constant $M \ge 0$ such that

$$d'(f(p), f(q)) \le M d(p, q)$$

for all $p, q \in E$.

Proposition 4. *Let $f \colon E \to E'$ be a Lipschitz map between metric space. Then f is continuous at all points of E.*

Problem 3. Prove this. *Hint:* Set $\delta = \dfrac{\varepsilon}{M}$. $\square$

Recall that on $\mathbb{R}^n$ we have defined the inner product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{n} a_j b_j$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. This was used to define the norm on $\mathbb{R}^n$ as

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

This in turn was used to define the distance function on $\mathbb{R}^n$ by

$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|.$$

Also recall that we have the Cauchy-Schwartz inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\| \|\mathbf{b}\|.$$

Problem 4. Let $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Define the function $f \colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

Show that f is continuous at all points of $\mathbb{R}^n$. *Hint:* Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ then show

$$f(\mathbf{p}) - f(\mathbf{q}) = \mathbf{a} \cdot (\mathbf{p} - \mathbf{q}).$$

Use the Cauchy-Schwartz inequality to show $|f(\mathbf{p}) - f(\mathbf{q})| \le \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\|$ and therefore f is Lipschitz with Lipschitz constant $M = \|\mathbf{a}\|$. $\square$

Problem 5. Define the functions $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x$ and $g(x, y) = y$. Show that f and g are continuous. *Hint:* As the two proofs are the same, it is enough to show that f is continuous. Let $\mathbf{a} = (1, 0)$ and $b = 0$ then $f(x, y) = (x, y) \cdot \mathbf{a}$ so one way to do this is to reduce it to the previous problem. $\square$

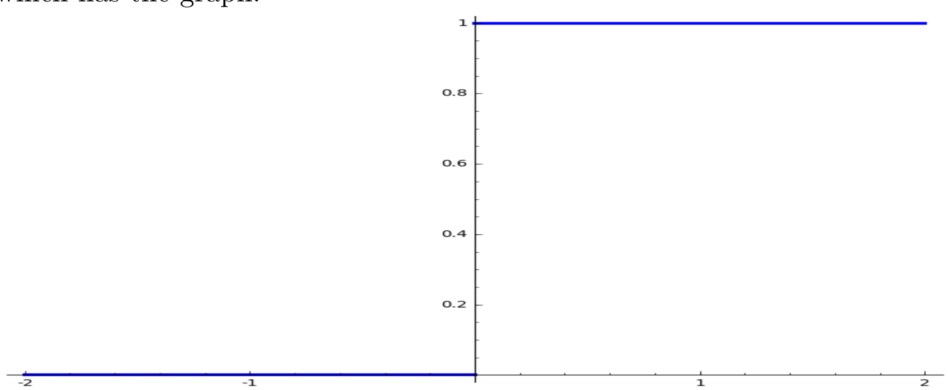
1.2. **Some examples of discontinuous functions.** We now give examples of some functions that are not continuous. We first record what it means for a function to not be continuous at a point.

Negation of Definition of Continuity. Let $f\colon E \to E'$ be a map between metric spaces. Let $p_0 \in E$. Then f is ***discontinuous*** at p_0 if and only if there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0) < \delta$ and $d'(f(p), f(p_0)) \geq \varepsilon$. $\square$

We now look at the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & 0 < x. \end{cases}$$

which has the graph:



We now show this is discontinuous at $x = 0$. Let $\varepsilon = 1/2$. Then for any $\delta > 0$ there is an $x > 0$ with $0 < x < \delta$. Then $x > 0$ and so $f(x) = 1$. As $f(0) = 0$ we have $|f(x) - f(0)| = |1 - 0| = 1 > \varepsilon$ as required.

Here is a more exotic example.

Problem 6. Define a function by

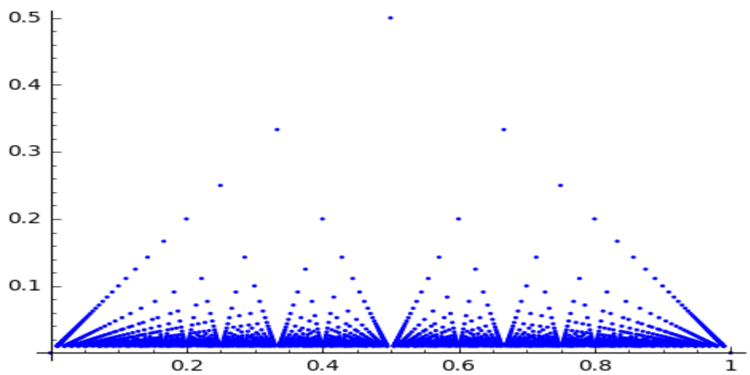
$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}; \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

That is $f(x)$ is zero with x is a rational number, and $f(x)$ is one when x is irrational. Show that f is discontinuous at all points of $\mathbb{R}$.

Problem 7 (Optional). Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ is rational in lowest terms;} \\ 0, & x \text{ is irrational.} \end{cases}$$

Here is the graph for rationals in $(0, 1)$ with denominators less than 100.



Show that f is continuous at all irrational points and discontinuous at all rational points. □

1.3. **Sums, products, and quotients of real valued continuous functions are continuous.**

Problem 8. Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}$$

then show f is continuous at $x = 1$.

Solution: We first note that

$$|f(x)-f(1)| = |\sqrt{x}-1| = \left|\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)}\right| = \left|\frac{x-1}{\sqrt{x}+1}\right| \le \left|\frac{x-1}{0+1}\right| = |x-1|.$$

Now let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then if $|x-1| < \delta$ implies

$$|f(x) - f(1)| \le |x-1| < \delta = \varepsilon$$

which is just what is needed to show that $f(x)$ is continuous at $x = 1$. □

Problem 9. Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}.$$

Show f is continuous at $x = a$ for any $a > 0$. □

Theorem 5. *Let E be a metric space and $f, g\colon E \to \mathbb{R}$ be functions and $c_1, c_2 \in \mathbb{R}$ constants. Assume f and g are continuous at p_0. Then*

(a) $c_1 f + c_2 g$ is continuous at p_0.
(b) The product fg is continuous at p_0.
(c) If $g(p_0) \ne 0$, then quotient $\dfrac{f}{g}$ is continuous at p_0.

Problem 10. (a) Prove part (a) of the Theorem.

(b) Prove part (b) of the Theorem. *Hint:* Note that by our standard adding and subtracting trick

$$\begin{aligned} |f(p)g(p) - f(p_0)g(p_0)| &= |f(p)g(p) - f(p)g(p_0) + f(p)g(p_0) - f(p_0)g(p_0)| \\ &\le |f(p)||g(p) - g(p_0)| + |f(p) - f(p_0)||g(p_0)| \end{aligned}$$

By the continuity of f there is a $\delta_1 > 0$ such that

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |f(p) - f(p_0)| < 1.$$

Show

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |f(p)| < |f(p_0)| + 1.$$

Again by the continuity of f there is a $\delta_2 > 0$ such that

$$d(p, p_0) < \delta_2 \quad \text{implies} \quad |f(p) - f(p_0)| < \frac{\varepsilon}{2|g(p_0| + 1}.$$

The continuity of g gives us a $\delta_3 > 0$ such that

$$d(p, p_0) < \delta_3 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{\varepsilon}{2(|f(p_0)| + 1)}$$

Now set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and show

$$d(p, p_0) < \delta \quad \text{implies} \quad |f(p)g(p) - f(p_0)g(p_0)| < \varepsilon$$

$\square$

Lemma 6. *Let E be a metric space and $g \colon E \to \mathbb{R}$ a function that is continuous at $p_0 \in E$ and with $g(p_0) \neq 0$. Then $\dfrac{1}{g}$ is also continuous at p_0.*

Problem 11. Prove this. *Hint:* As g is continuous at p_0 and $g(p_0) \neq 0$, there is a $\delta_1 > 0$ such that

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{|g(p_0)|}{2}.$$

Use this to show

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad \frac{1}{|g(p)|} < \frac{2}{|g(p_0)|},$$

and therefore

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad \left| \frac{1}{g(p)} - \frac{1}{g(p_0)} \right| \leq \frac{2|g(p_0) - g(p)|}{|g(p_0)|^2}$$

The continuity of g at p_0 implies there is a $\delta_2 > 0$ such that

$$d(p, p_0) < \delta_2 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{|g(p_0)|^2 \varepsilon}{2}.$$

And you should be able to take it from here. $\square$

Problem 12. Use Lemma 6 and part (b) of Theorem 5 to prove part (c) of Theorem 5. $\square$

Proposition 7. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be the polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Then f is continuous at all points of $\mathbb{R}$.

Problem 13. Prove this. *Hint:* Probably the easiest way is by induction on n. The base of the induction is $n = 0$ in which case $f(x) = a_0$ is a constant which is clearly continuous. Or we can use the base case of $n = 1$ in which case $f(x) = a_1 x + a_0$ is Lipschitz and therefore continuous.

Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that all polynomials of degree 4 are continuous and let

$$f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

be a polynomial of degree 5. Write it as

$$\begin{aligned} f(x) &= x(a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1) + a_0 \\ &= x g(x) + a_0 \end{aligned}$$

where $g(x) = a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x^1 + a_1$ is a polynomial of degree 4. By the induction hypothesis $g(x)$ is continuous and the function x is continuous. Whence f is of the form

$$f = (\text{continuous function}) \times (\text{continuous function}) + (\text{constant})$$

and therefore f is continuous. Use this idea to do the general induction step. $\square$

Theorem 8. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be maps between metric spaces. Let $p_0 \in E$ and assume that f is continuous at p_0 and g is continuous at $f(p_0)$. Then the composition $g \circ f$ is continuous at p_0.*

Problem 14. Prove this. *Hint:* Let $\varepsilon > 0$. By the definition of g being continuous at $f(p_0)$ there is a $\delta_1 > 0$ such that $d(q, f(p_0)) < \delta_1$ implies $d(g(q), g(f(p_0))) < \varepsilon$. By the definition of f being continuous at p_0 there is $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0)) < \delta_1$. Now show $d(p, p_0) < \delta$ implies $d\big((g \circ f)(p), (g \circ f)(p_0)\big) < \varepsilon$. $\square$

1.4. **Conditions equivalent to a function being continuous at a point.** We are going to give other conditions that imply a function is continuous, but first we review a bit of set theory from the beginning of the term.

Let $f\colon E \to E'$ be a map between sets. Recall that if $A \subseteq E$, then the ***image*** of A under f is

$$f[S] = \{f(x) : x \in A\}.$$

And if $B \subseteq E'$ the ***preimage*** of B under f is

$$f^{-1}[B] = \{x \in E : f(x) \in B\}.$$

We recall that taking preimages behaves well with respect to taking unions and intersections.

Proposition 9. *Let $f\colon E \to E'$ be a map between sets and let $\{S_\alpha\}_{\alpha\in I}$ be a collections of subsets of E'. (That is for each $\alpha \in A$ the $S_\alpha \subseteq E'$.) Then*

$$f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] = \bigcup_{\alpha\in I} f^{-1}[S_\alpha] \qquad \textit{and}$$
$$f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[S_\alpha],$$

Proof. To prove the first equality:

$$\begin{aligned} x \in f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcup_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for at least one } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for at least one } \alpha \in I \\ &\iff x \in \bigcup_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

This shows that $f^{-1}\Big(\bigcup_{\alpha\in I} S_\alpha\Big)$ and $\bigcup_{\alpha\in I} f^{-1}(S_\alpha)$ have the same elements and therefore are equal.

Likewise

$$\begin{aligned} x \in f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcap_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for all } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for all } \alpha \in I \\ &\iff x \in \bigcap_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

and therefore $f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[S_\alpha]$. ∎

We recall that in the book's notation if S is a subset of some set E then the ***compliment*** of S in E is

$$\mathcal{C}(S) = \{x \in E : x \notin S\}.$$

That is $\mathcal{C}(S)$ is the set of points of E that are not in S. Taking compliments is also well behaved with respect to taking preimages.

Proposition 10. *Let $f\colon E \to E'$ be a map between sets and let $S \subseteq E'$. Then*

$$f^{-1}[\mathcal{C}(S)] = \mathcal{C}(f^{-1}[S]).$$

(Here $\mathcal{C}(S)$ is the compliment of S in E' and $\mathcal{C}(f^{-1}[S])$ is the compliment of $f^{-1}[S]$ in E.)

Proof. For $x \in E$ we have

$$\begin{aligned} x \in \mathcal{C}(f^{-1}[S]) &\iff x \in f^{-1}[S] \\ &\iff f(x) \notin S \\ &\iff f(x) \in \mathcal{C}(S) \\ &\iff x \in f^{-1}[\mathcal{C}(S)]. \end{aligned}$$

Therefore $\mathcal{C}(f^{-1}[S])$ and $f^{-1}[\mathcal{C}(S)]$ have the same elements and thus are equal. □

We summarize this as

Taking preimages preserves unions, intersections, and compliments.

Theorem 11. *Let $f \colon E \to E'$ be a function between metric spaces and let p_0. Then the following conditions are equivalent:*

(a) f is continuous at p_0.

(b) f does the right thing to sequences converging to p_0. Explicitly: If $p_1, p_2, p_3, \ldots$ is a sequence in E with $\lim_{n\to\infty} p_n = p_0$, then

$$\lim_{n\to\infty} f(p_n) = f(p_0).$$

(c) For any open ball $B(f(p_0), r)$ about $f(p_0)$, the preimage $f^{-1}\big[B(f(p_0), r)\big]$ contains a open ball about p_0. (That is there is a $\delta > 0$ such that $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), r)\big]$.)

Problem 15. In Theorem 11 prove (a) implies (b). *Hint:* Let $\varepsilon > 0$. Then the continuity of f at p_0 implies there is a $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0))$. Let $\lim_{n\to\infty} p_n = p_0$. Then there is a N such that $n \geq N$ implies $d(p_n, p_0) < \delta$. Now show that $n \geq N$ implies $d(f(p_n), f(p_0)) < \varepsilon$. □

Problem 16. In Theorem 11 prove (b) implies (a). *Hint:* It is easier to prove the contrapositive: $\sim$(a) implies $\sim$(b). That is we assume that (a) is false and prove that (b) is false. If (a) is false there is $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0)\delta$ and $d(f(p), f(p_0)) \geq \varepsilon$. Letting $\delta = 1/n$ shows there is a point p_n with $d(p_n, p_0) < 1/n$ and $d(f(p_n), f(p_0)) \geq \varepsilon$. Show that $\lim_{n\to\infty} p_n = p_0$, but $\lim_{n\to\infty} f(p_n) \neq f(p_0)$. □

Problem 17. In Theorem 11 prove (a) implies (c). *Hint:* Assume that f is continuous at p_0 and let $r > 0$. Letting $\varepsilon = r$ in the definition of f being continuous at p_0 gives a $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0)) < r$. If $p \in B(p_0, \delta)$, then $d(p, p_0) < \delta$ and therefore $d(f(p), f(p_0)) < r$. That is $p \in B(p_0, \delta)$ implies $f(p) \in B(f(p_0), r)$. Now use the definition of the preimage $f^{-1}\big[B(f(p_0), r)\big]$ to show this implies $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), r)\big]$. □

Problem 18. In Theorem 11 prove (c) implies (a). *Hint:* Assume that (c) holds and let $\varepsilon > 0$. Then in the statement of (c) let $r = \varepsilon$. As (c) holds, then is $\delta > 0$ such that $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), \varepsilon)\big]$. Let $p \in E$ with $d(p_0, p) < \delta$, then $p \in B(p > \delta) \subseteq f^{-1}\big[B(f(p_0), \varepsilon)\big]$. Then, by the definition of the $f^{-1}\big[B(f(p_0), \varepsilon)\big]$, this implies $f(p) \in B(f(p_0), \varepsilon)$. And you should be able to finish from here. $\square$

Proof of Theorem 11. Combining Problems 15, 16, 17, and 18 we have

$$(b) \iff (a) \iff (c)$$

which shows the three conditions are equivalent. $\square$

1.5. **Conditions equivalent to a function being continuous at all points.**

Definition 12. A map $f\colon E \to E'$ is ***continuous*** if and only if it is continuous at every point of E. $\square$

Theorem 13. *Let $f\colon E \to E'$ be a map between metric spaces. Then the following are equivalent.*

(a) f is continuous.
(b) f does the right thing to convergent sequences in E. That is if $\langle p_n \rangle_{n=1}^{\infty}$ is a convergent sequence in E, then

$$\lim_{n\to\infty} f(p_n) = f\big(\lim_{n\to\infty} p_n\big).$$

(c) If U is an open set in E', then the preimage $f^{-1}[U]$ is an open set in E. (That is preimages of open sets are open.)
(d) If F is a closed set in E', then the preimage $f^{-1}[F]$ is a closed set in E. (That is the preimages of closed sets are closed.)

Problem 19. Prove this. *Hint:* Note that Theorem 11 easily implies

$$(a) \iff (b) \iff (c)$$

and you can assume these equivalences. So all you have to do is prove (c) $\iff$ (d). $\square$

Now let us do some practice in using these equivalences.

Proposition 14. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be continuous maps between metric spaces. Then the composition $g \circ f\colon E \to E''$ is continuous.*

Problem 20. This is a special case of Theorem 8. In Problem 14 you gave a ε-δ proof. Now give a proof using that continuity is equivalent to doing the right thing by convergent sequences. *Hint:* It is enough to show that if f and g are continuous, and $\lim_{n\to\infty} p_n = p_0$ in E that $\lim_{n\to\infty} g(f(p_n)) = g(f(p_0))$ in E''. $\square$

Lemma 15. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be functions. Show that for and $S \subseteq E''$ that*

$$(g \circ f)^{-1}[S] = f^{-1}\big[g^{-1}[S]\big].$$

$\square$

Problem 21. Give anther proof of Proposition 14 using that continuity is equivalent to the preimages of open sets being open. *Hint:* Let f and g be continuous and $U \subseteq E''$ be open. All you need to show is that $(g \circ f)^{-1}[U]$ is open and the last lemma should make this easy.