CONSTRUCTION OF THE REAL NUMBERS

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Contents

In the presentation of the real numbers, $\mathbb{R}$, we have given in class is that $\mathbb{R}$ is an ordered field that stratifies the least upper bound axiom. But we have not shown that such a field exists. Here we construct the reals by starting with the rational numbers, $\mathbb{Q}$, "completing" it by adjoining in all the limits of Cauchy sequences of rational numbers. This construction of the real numbers is due to Cantor and was published in 1883. The first published rigorous construction of the reals was Dedekind in 1872. Much earlier, in 1585, the Flemish mathematician Simon Stevin defined the set of decimal numbers, which is the same as the set of real numbers. But no proof of the of any of the basic properties of the reals was given. And while working with decimals is psychologically natural, the details[1] are even messier than the construction given here. For a nice discussion of the different constructions of the reals see the survey *The Real Numbers - A Survey of Constructions* by Ittay Weiss at `https://arxiv.org/pdf/1506.03467`.

1. Existence of of an complete ordered field.

We will use the notation $\langle p_n \rangle_{n=1}^{\infty}$ for a sequence

$$\langle p_n \rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots \rangle.$$

Definition 1. The sequence $\langle p_n \rangle_{n=1}^{\infty}$ of rational numbers is a ***Cauchy sequence*** if and only if for all rational numbers $\varepsilon > 0$, there is a natural number N such that

$$m, n \geq N \quad \text{implies} \quad |p_m - p_n| < \varepsilon.$$

□

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[1]And the part that might seem easiest at first is quite tricky. For example given two infinite decimals $a = .a_1a_2a_3a_4\ldots$ and $b = .b_1b_2b_3b_4\ldots$ between 0 and 1, give a formula for the sum $a + b$. And then try to prove this is associative.

Let

$$\mathcal{R} := \{\langle p_n\rangle_{n=1}^{\infty} : \text{where } \langle p_n\rangle_{n=1}^{\infty} \text{ is a Cauchy sequence in } \mathbb{Q}.\}$$

We will denote elements of $\mathcal{R}$ by with the abbreviation $\langle p\rangle$.

Define an relation $\sim$ on $\mathcal{R}$ by

$$\langle p_n\rangle_{n=1}^{\infty} \sim \langle q_n\rangle_{n=1}^{\infty} \quad \text{if and only if} \quad \lim_{n\to\infty}(p_n - q_n) = 0.$$

More explicitly this means that for all rational numbers $\varepsilon > 0$ there is a natural number N such that

$$n \geq N \quad \text{implies} \quad |p_n - q_n| < \varepsilon.$$

Lemma 2. *The relation $\sim$ on $\mathcal{R}$ is an **equivalence relation**. That is*

*(a) $\sim$ is **reflective**, that is for any element, $\langle p\rangle$, of $\mathcal{R}$ we have $\langle p\rangle \sim \langle p\rangle$.*
*(b) $\sim$ is **symmetric**. That is for all $\langle p\rangle, \langle q\rangle \in \mathcal{R}$*

$$\langle p\rangle \sim \langle q\rangle \quad \textit{implies} \quad \langle q\rangle \sim \langle p\rangle.$$

*(c) $\sim$ is **transitive**. That is for $\langle p\rangle, \langle q\rangle, \langle r\rangle \in \mathcal{R}$*

$$\langle p\rangle \sim \langle q\rangle \quad \textit{and} \quad \langle q\rangle \sim \langle r\rangle \quad \textit{implies} \quad \langle p\rangle \sim \langle r\rangle.$$

□

Problem 1. Prove this. *Hint:* Parts (a) and (b) follow pretty much directly from the definition. For part (c) note by use of the triangle inequality

$$|p_n - r_n| \leq |p_n - q_n| + |q_n - r_n|.$$

Therefore showing $\sim$ is transitive is standard $\varepsilon/2$ argument. □

We can add and multiply elements of $\mathcal{R}$ in a natural way. That is

$$\langle p_n\rangle_{n=1}^{\infty} + \langle q_n\rangle_{n=1}^{\infty} = \langle p_n + q_n\rangle_{n=1}^{\infty}$$

and

$$\langle p_n\rangle_{n=1}^{\infty}\langle q_n\rangle_{n=1}^{\infty} = \langle p_n q_n\rangle_{n=1}^{\infty}.$$

Or in slightly different notation addition is

$$\langle p_1, p_2, p_3, \ldots\rangle + \langle q_1, q_2, q_3, \ldots\rangle = \langle p_1 + q_1, p_2 + q_2, p_3 + q_3, \ldots\rangle$$

and multiplication is

$$\langle p_1, p_2, p_3, \ldots\rangle\langle q_1, q_2, q_3, \ldots\rangle = \langle p_1 q_1, p_2 q_2, p_3 q_3, \ldots\rangle.$$

Lemma 3. *The set $\mathcal{R}$ is closed under addition. That is if $\langle p\rangle, \langle q\rangle \in \mathcal{R}$, then $\langle p\rangle + \langle q\rangle \in \mathcal{R}$.*

Problem 2. Prove this. *Hint:* Note

$$|(p_m + q_m) - (p_n + q_n)| \leq |p_m - p_n| + |q_m - q_n|$$

and therefore a $\varepsilon/2$ argument will work. □

Lemma 4. *If $\langle p\rangle, \langle q\rangle, \langle x\rangle, \langle y\rangle \in \mathcal{R}$ with*

$$\langle p\rangle \sim \langle q\rangle \quad \textit{and} \quad \langle x\rangle \sim \langle y\rangle$$

then

$$\langle p\rangle + \langle x\rangle \sim \langle q\rangle + \langle y\rangle.$$

Problem 3. Prove this. *Hint:* Note

$$|(p_n + x_n) - (q_n + y_n)| \le |p_n - q_n| + |x_n - y_n|$$

and so this is anther $\varepsilon/2$ proof. □

Lemma 5. *If $\langle p\rangle \in \mathcal{R}$, then there is a $B \in \mathbb{Q}$ such that $|p_n| \le B$ for all n.*

Problem 4. Prove this. *Hint:* Let $\varepsilon = 1$ in the definition of $\langle p\rangle$ being Cauchy. Then there is a natural number N such that

$$m, n \ge N \quad \text{implies} \quad |p_m - p_n| < \varepsilon = 1.$$

Show

$$n \ge N \quad \text{implies} \quad |p_n| \le |p_N| + 1$$

and therefore

$$B = \max\{|p_1|, |p_2|, \dots, |p_{N-1}|, |p_N| + 1\}.$$

works as the required bound. □

Lemma 6. *The set $\mathcal{R}$ is closed under multiplication.*

Problem 5. Prove this. *Hint:* We need to show that if $\langle p\rangle, \langle q\rangle \in \mathcal{R}$, then $\langle p\rangle\langle q\rangle \in \mathcal{R}$. Note

$$\begin{aligned}
|p_m q_m - p_n q_n| &= |p_m q_m - p_m q_n + p_m q_n - p_n q_n| \\
&= |p_m(q_m - q_n) + (p_m - p_n)q_n| \\
&\le |p_m||q_m - q_n| + |p_m - q_n||q_n|
\end{aligned}$$

By Lemma 5 there are $B_1, B_2 \in \mathbb{Q}$ such that

$$|p_n| \le B_1 \quad \text{and} \quad |q_n| \le B_2 \quad \text{for all} \quad n \in \mathbb{N}.$$

Therefore

$$|p_m q_m - p_n q_n| \le B_1|q_m - q_n| + |p_m - p_n|B_2$$

and thus a $\varepsilon/(2B_1)$, $\varepsilon/(2B_2)$ proof will work. □

Lemma 7. *If $\langle p\rangle, \langle q\rangle, \langle x\rangle, \langle y\rangle \in \mathcal{R}$ with*

$$\langle p\rangle \sim \langle q\rangle \quad \textit{and} \quad \langle x\rangle \sim \langle y\rangle$$

then

$$\langle p\rangle\langle x\rangle \sim \langle q\rangle\langle y\rangle.$$

Problem 6. Prove this. *Hint:* This is much like that last proof. Our basic adding and subtracting trick together with the triangle inequality gives

$$|p_n x_n - q_n y_n| \le |p_n||x_n - y_n| + |p_n - q_n||y_n|.$$

By Lemma 5 there are rational numbers $B_1, B_2 > 0$ such that

$$|p_n| \le B_1 \quad \text{and} \quad |y_n| \le B_2 \quad \text{for all} \quad n \in \mathbb{N}.$$

and you should be able to take it form here. □

We can now define the our candidate for the real numbers. If $\langle p \rangle \in \mathcal{R}$ let

$$[\langle p \rangle] = \{\langle q \rangle : \langle q \rangle \in \mathcal{R} \text{ and } \langle q \rangle \sim \langle p \rangle\}$$

be the equivalence class of $\langle p \rangle$ under the equivalence relation $\sim$. A restatement of Lemma 4 is that if we define addition of equivalence classes by

$$[\langle p \rangle] + [\langle x \rangle] = [\langle p \rangle + \langle z \rangle]$$

then this operation is well defined. Likewise we can define define multiplication of equivalence classes by

$$[\langle p \rangle][\langle q \rangle] = [\langle p \rangle \langle q \rangle]$$

and Lemma 7 implies this is well defined.

Definition 8. Define the ***real numbers*** to be the set

$$\mathbb{R} = \{[\langle p \rangle] : \langle p \rangle \in \mathcal{R}\}$$

of equivalence classes of $\mathcal{R}$ under the equivalence relation $\sim$. Addition and multiplication in $\mathbb{R}$ are defined by

$$[\langle p \rangle] + [\langle q \rangle] = [\langle p \rangle + \langle q \rangle] \quad \text{and} \quad [\langle p \rangle][\langle q \rangle] = [\langle p \rangle \langle q \rangle].$$

□

The notation $[\langle p \rangle]$ is too ugly to keep using, so let us shorten it to

$$[p] = [\langle p \rangle].$$

Theorem 9. *With these operations the set $\mathbb{R}$ is a **ring** in the sense of abstract algebra. That is*

(a) *Addition and multiplication are **commutative**: for all $[p], [q] \in \mathbb{R}$*

$$[p] + [q] = [q] + [p] \quad \textit{and} \quad [p][q] = [q][p].$$

(b) *Addition and multiplication are **associative**: for all $[p], [q], [r] \in \mathbb{R}$*

$$\big([p] + [q]\big) + [r] = [p] + \big([q] + [r]\big)$$
$$\big([p][q]\big)[r] = [p]\big([q][r]\big).$$

(c) *Multiplication **distributes** over addition: for a all $[p], [q], [r] \in \mathbb{R}$*

$$[p]\big([q] + [r]\big) = [p][q] + [p][q].$$

(d) *The element $[0] = [\langle 0, 0, 0, \ldots \rangle]$ is an **additive identity**: for all $[p] \in \mathbb{R}$*

$$[0] + [p] = [p] + [0] = [p].$$

(e) *The element* $[1] = [\langle 1, 1, 1, \ldots \rangle]$ *is a* ***multiplicative identity****: for all* $[\langle p \rangle] \in \mathbb{R}$

$$[1][p] = [p][1] = [p]$$

Problem 7. Convince yourself this is true. Which is to say that you should write out enough of the proof (which is only definition chasing) that you feel that you could do it on a quiz if ask to do so. $\square$

We want $\mathbb{R}$ to be a field. Theorem 9 does almost all the work in showing this to be the case. What is missing is that we need to show that non-zero elements of $\mathbb{R}$ have multiplicative inverses. It is easy to guess what these should be: if

$$[p] = [\langle p_1, p_2, p_3, p_4, \ldots \rangle]$$

is not the zero element of $\mathbb{R}$ then the multiplicative inverse should be

$$\frac{1}{[p]} = \left[\left\langle \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4} \ldots \right\rangle \right].$$

The problem with this is that it assumes that all the p_ns have $p_n \neq 0$ so that $1/p_n$ is defined. But even when $[p] \neq [0]$, this does not mean that all the p_n's are not zero. For example if

$$\langle p \rangle = \langle 0, 0, 1, 1, 1, 1, \ldots \rangle$$

(all the rest of the elements are 1), then $1/[p]$ as defined above does not make sense. But

$$\langle p \rangle \sim \langle 1 \rangle = \langle 1, 1, 1, 1, \ldots \rangle.$$

and so $[p] = [1]$ and thus $[p]$ is its own inverse.

The following will let us take care of the problem of having elements $p_n = 0$ in $\langle p \rangle = \langle p_1, p_2, p_3, \ldots \rangle$.

Lemma 10. *Let* $\langle p \rangle, \langle \tilde{p} \rangle \in \mathcal{R}$ *such that* $p_n = \tilde{p}_n$ *for all but finitely many* n*. Then* $\langle p \rangle \sim \langle \tilde{p} \rangle$*.*

Problem 8. Prove this. *Hint:* That assumption that $p_n = \tilde{p}_n$ for all but finitely many n implies that is a natural number N such that $n \geq N$ implies $p_n = \tilde{p}_n$. That is $n \geq N$ implies $|p_n - \tilde{p}_n| = 0$. $\square$

Lemma 11. *Let* $\langle p \rangle \in \mathcal{R}$ *with* $\langle p \rangle \not\sim \langle 0 \rangle$ *(that is* $\lim_{n\to\infty} p_n \neq 0$*). Then there is a natural number* N *and a rational number* $\rho > 0$ *such that*

$$n \geq N \quad \textit{implies} \quad |p_n| \geq \rho.$$

Problem 9. Prove this. *Hint:* Here is an outline of a proof by contradiction. If it is false, then for any natural number N and any rational number $\rho > 0$ there is a natural number $k = k(N, \rho)$ (that is it depends on both N and ρ) such that

$$k(N, \rho) \geq N \quad \text{and} \quad |p_{k(N,\rho)}| < \rho.$$

Let $\varepsilon > 0$. Because $\langle p \rangle$ is a Cauchy sequence, there is a natural number N such that

$$m, n \geq N \quad \text{implies} \quad |p_m - p_n| < \frac{\varepsilon}{2}.$$

The natural number $k(N, \varepsilon/2)$ satisfies

$$k(N, \varepsilon/2) \geq N \quad \text{and} \quad |p_{k(N,\varepsilon/2)}| < \frac{\varepsilon}{2}.$$

The triangle inequality and the adding and subtracting trick gives

$$|p_n| \leq |p_{k(N,\varepsilon/2)}| + |p_n - p_{(k(N,\varepsilon/2)}|.$$

Use these inequalities to show

$$n \geq N \quad \text{implies} \quad |p_n| < \varepsilon$$

which contradicts $\lim_{n\to\infty} p_n \neq 0$. □

If $\langle p \rangle = \langle p_n \rangle_{n=1}^{\infty} \in \mathcal{R}$ let $\langle p^* \rangle$ be the element $\langle p_n^* \rangle_{n=1}^{\infty}$ where

$$p_n^* = \begin{cases} p_n, & p_n \neq 0; \\ 1, & p_n = 0. \end{cases}$$

Lemma 12. *Let $\langle p \rangle \in \mathcal{R}$ with $\langle p \rangle \not\sim 0$. Then*

(a) $p_n = p_n^*$ for all but finitely many n,
(b) $\langle p \rangle \sim \langle p^* \rangle$, and
(c) there is a rational number $\delta > 0$ such that

$$|p_n^*| \geq \delta \quad \textit{for all} \quad n \in \mathbb{N}.$$

Problem 10. Prove this. *Hint:* For (a) use Lemma 11, and then use Lemma 8 to show (b). For (c) Use Lemma 11 to find N and $\rho > 0$ such that $n \geq N$ implies $|p_n| \geq \rho$. Set

$$\delta = \min\{|p_1^*|, |p_2^*|, \ldots, |p_{N-1}^*|, \rho\}$$

show $|p_n^*| \geq \delta$ for all n and $\delta > 0$. □

Lemma 13. *Let $\langle p \rangle \in \mathcal{R}$ with $\langle p \rangle \not\sim \langle 0 \rangle$. Then*

$$\frac{1}{\langle p^* \rangle} = \left\langle \frac{1}{p_n^*} \right\rangle_{n=1}^{\infty} \in \mathcal{R}.$$

Problem 11. Prove this. *Hint:* By Lemma 12 there is a $\delta > 0$ such that

$$|p_n^*| \geq \delta \quad \text{for all} \quad n \in \mathbb{N}.$$

Use this to show

$$\left| \frac{1}{p_m^*} - \frac{1}{p_n^*} \right| \leq \frac{|p_n^* - p_m^*|}{\delta^2}$$

and use this to show $1/\langle p^* \rangle$ is a Cauchy sequence. □

Lemma 14. *Let $\langle p \rangle, \langle q \rangle \in \mathcal{R}$ with $\langle p \rangle \sim \langle q \rangle$ and $\langle p \rangle \not\sim \langle 0 \rangle$. Then*

$$\frac{1}{\langle p^* \rangle} \sim \frac{1}{\langle q^* \rangle}.$$

Problem 12. Prove this. *Hint:* By Lemma 12 there are $\delta_1, \delta_2 > 0$ such that

$$|p_n^*| \geq \delta_1 \quad \text{and} \quad |q_n^*| \geq \delta_2 \quad \text{for all} \quad n \in \mathbb{N}$$

Show

$$\left| \frac{1}{p_n^*} - \frac{1}{q_n^*} \right| \leq \frac{|q_n^* - p_n^*|}{\delta_1 \delta_2}$$

and use this to show

$$\lim_{n \to \infty} \left(\frac{1}{p_n^*} - \frac{1}{q_n^*} \right) = 0.$$

and therefore $1/\langle p^* \rangle \sim 1/\langle q^* \rangle$. □

Theorem 15. *The set* $\mathbb{R} = \mathcal{R}/\sim$ *is a field.*

Problem 13. Prove this. *Hint:* By Theorem 9, $\mathbb{R}$ satisfies all the axioms for a field other than the existence of multiplicative inverses. You should be able to use the last several lemmas to show that if $[p] \in \mathbb{R}$ and $[p] \neq [0]$, then $[p]$ has a well defined multiplicative inverse. □

Now that we have that $\mathbb{R}$ is a field we need to show that it is ordered. That is we need to define the set of positive elements.

Definition 16. Let $\langle p \rangle \in \mathcal{R}$. Then

(a) $\langle p \rangle$ is ***eventually positive*** if and only if there is a yrational number $\rho > 0$ and a natural number N such that

$$n \geq N \quad \text{implies} \quad p_n \geq \rho.$$

(b) $\langle p \rangle$ is ***eventually negative*** if and only if rational number $\rho > 0$ and a natural number N such that

$$n \geq N \quad \text{implies} \quad p_n \leq -\rho.$$

Lemma 17. *Let* $\langle p \rangle \in \mathcal{R}$ *with* $\langle p \rangle \not\sim \langle 0 \rangle$. *Then* $\langle p \rangle$ *is either eventually positive or eventually negative.*

Problem 14. Prove this. *Hint:* By Lemma 11 there is a N_1 and a $\rho > 0$ such that

$$n \geq N_1 \quad \text{implies} \quad |p_n| \geq \rho.$$

Therefore if $n \geq N_1$ we have that either $p_n \geq \rho$ or $p_n \leq -\rho$. Thus to finish we only need show that for all $n \geq N_1$ that all the p_n have the same sign (i.e. are all positive or all negative).

As $\langle p \rangle \in \mathcal{R}$ implies $\langle p \rangle$ is a Cauchy sequence, thus there is N_2 such that

$$m, n \geq N_2 \quad \text{implies} \quad |p_m - p_n| < \rho.$$

Let $N = \max\{N_1, N_2\}$ and assume, towards a contradiction, there are $m, n \geq N$ such that p_m and p_n have opposite signs. Show that this implies

$$|p_m - p_n| \geq 2\rho$$

which gives a contradiction. □

Lemma 18. *If $\langle p\rangle \in \mathcal{R}$ is eventually positive and $\langle q\rangle \sim \langle p\rangle$, then $\langle q\rangle$ is eventually positive.*

Problem 15. Prove this. $\square$

We now define the set of positive elements of $\mathbb{R}$ as

$$\mathbb{R}_+ = \{[p] : \langle p\rangle \text{ is eventually positive}\}.$$

Theorem 19. *The field $\mathbb{R}$ with $\mathbb{R}_+$ as its set of positive elements is an ordered field.*

Problem 16. Prove this. *Hint:* All that needs to be done is show that the positivity axioms hold. That is

Pos 1 The set $\mathbb{R}_+$ is closed under addition and multiplication.

Pos 2 The ***trichotomy principle*** holds. That is for any $[p] \in \mathbb{R}$ exactly one of

$$\begin{aligned} [p] &\in \mathbb{R}_+ \\ [p] &= [0] \\ -[p] &\in \mathbb{R}_+ \end{aligned}$$

holds. Lemma 17 is useful here. $\square$

Then, just as we have done for other ordered fields, define

$$[p] < [q] \quad \text{if and only if} \quad [q] - [p] \in \mathbb{R}_+,$$

and the ***absolute value*** as

$$\Big|[p]\Big| = \begin{cases} [p], & [p] > [0]; \\ [0], & [p] = [0] \\ -[p], & [p] < [0]. \end{cases}$$

The following is obvious, but doing the proof is good practice with using the definitions.

Lemma 20. *Let $\langle p\rangle = \langle p_n\rangle_{n=1}^{\infty}, \langle q\rangle = \langle q_n\rangle_{n=1}^{\infty} \in \mathcal{R}$ with $p_n \leq q_n$ for all $n \in \mathbb{N}$. Then $[p] \leq [q]$.*

Problem 17. Prove this. *Hint:* If this is false, then by the trichotomy principle $[p] > [q]$ and thus $[p - q] > 0$. This implies that $\langle p_n - q_n\rangle_{n=1}^{\infty}$ is is eventually negative which can be used to get a contradiction. $\square$

Problem 18. In light of the last lemma it is tempting to conclude that if in the hypothesis of Lemma 20 that $p_n < q_n$ for all n implies $[p] < [q]$. Give an example to show this is false. *Hint:* If $p_n = 1/n$ for all n show that $[p] = [0]$. $\square$

We let $r \in \mathbb{Q}$ be a rational number. Then we can view r as an element of $\mathbb{R}$ as

$$r = [r] = [\langle r, r, r, r, \ldots\rangle]$$

where $\langle r, r, r, r, \ldots\rangle$ is the constant sequence.

We still have to prove the least upper bound axiom holds in $\mathbb{R}$. To start we show that every element of $\mathbb{R}$ is close to a rational number.

Lemma 21. *Let $\langle p\rangle = \langle p_1, p_2, p_3, \ldots\rangle \in \mathcal{R}$ and let $\varepsilon > 0$ be a positive real number. Then there is a natural number N such that*

$$n \geq N \quad \textit{implies} \quad \Big|[p] - p_n\Big| < \varepsilon$$

Problem 19. Prove this. *Hint:* As $\langle p\rangle$ is a Cauchy sequence there is natural number N such that

$$m, n \geq N \quad \text{implies} \quad |p_m - p_n| < \frac{\varepsilon}{2}.$$

Then for $n \geq N$

$$p_n = p_N + (p_n - p_N) < p_N + \frac{\varepsilon}{2}.$$

This means that if we identify $p_N + \varepsilon/2$ with the constant sequence $\langle p_N + \varepsilon/2, p_N + \varepsilon/2, p_N + \varepsilon/2, \ldots\rangle$ then the sequence

$$p_N + \frac{\varepsilon}{2} - \langle p\rangle$$

is eventually positive. This implies $p_N + \frac{\varepsilon}{2} - [p] > 0$ and thus

$$[p] < p_N + \frac{\varepsilon}{2}.$$

For $n \geq N$ we have

$$p_N = p_n + (P_N - p_n) < p_n + \frac{\varepsilon}{2}.$$

Combining these inequalities gives

$$[p] < p_N + \frac{\varepsilon}{2} < p_n + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = p_n + \varepsilon.$$

Do a similar argument to show that for $n \geq N$ the inequality

$$p_n - \varepsilon < [p].$$

and therefore the required implication

$$n \geq N \quad \text{implies} \quad \Big|[p] - p_n\Big| < \varepsilon$$

holds. □

Lemma 22. *Let $[p] \in \mathbb{R}$ and $\varepsilon > 0$ be a rational number. Then there is a rational number r such that*

$$\Big|[p] - [r]\Big| < [\varepsilon].$$

Problem 20. Prove this as a corollary to Lemma 21. □

Proposition 23. *The axiom of Archimedes holds in $\mathbb{R}$. That is if $[p] \in \mathbb{R}$ then there is a natural number n with $n > [p]$.*

Problem 21. Prove this. *Hint:* If $[p] \leq 0$, just use $n = 1$. This result is known to be true for the rational numbers and you are allowed to assume that for any rational number $r > 0$, there is a natural number n_1 with $r < n_1$. Let $[p] > 0$. By Lemma 22 there is a rational number r such that $\Big|[p]-r\Big| < 1$. Thus there is a natural number n_1 with $r < n_1$. Let $n = n_1+1$ and show $[p] < n$. $\square$

Lemma 24. *Let $[p], [q] \in \mathbb{R}$ with $[p] < [q]$. There there is a rational number r with $[p] < r < [q]$.*

Problem 22. Prove this. *Hint:* By Lemma 23 there is a natural number n such that

$$n > \frac{2}{[q]-[p]}.$$

This implies

$$\frac{1}{n} < \frac{[q]-[p]}{2}.$$

By Lemma 22 there are rational numbers r_1 and r_2 with

$$\Big|[p]-r_1\Big| < \frac{1}{n} \quad \text{and} \quad \Big|[q]-r_2\Big| < \frac{1}{n}.$$

Show this implies the inequalities

$$[p] - \frac{[q]-[p]}{2} < r_1 < [p] + \frac{[q]-[p]}{2} \tag{1}$$

$$[q] - \frac{[q]-[p]}{2} < r_2 < [q] + \frac{[q]-[p]}{2}. \tag{2}$$

Let r be the rational number

$$r = \frac{r_1+r_2}{2}.$$

and add the two inequalities (1) and (2) to show

$$[p] < r < [q]$$

to complete the proof. $\square$

Lemma 25. *Let $\langle r\rangle = \langle r_1, r_2, r_3, \ldots\rangle \in \mathcal{R}$. Let $[s] \in \mathbb{R}$ such that for all $n \in N$ the inequality*

$$r_n \leq [s]$$

holds. Then

$$[r] \leq [s].$$

Problem 23. Prove this. *Hint:* Let $\varepsilon > 0$ be a rational number. By Lemma 21 there is a natural number N such that

$$n \geq N \quad \text{implies} \quad \Big|[r]-r_n\Big| < \varepsilon.$$

In particular this implies

$$\begin{aligned}[r] &= r_N + ([r] - r_N) \\ &< r_N + \varepsilon \\ &\leq [s] + \varepsilon.\end{aligned}$$

Thus for every rational number $\varepsilon > 0$ we have

$$[r] - [s] < \varepsilon.$$

Towards a contradiction assume that $[r] > [s]$. Then use Lemma 24 to find a rational number $\varepsilon > 0$ with $[0] < \varepsilon < [r] - [s]$ and to get a contradiction. $\square$

Lemma 26. *Let $S \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then for each natural number n there is an $[s_n] \in S$ such that*

$$[s_n] + \frac{1}{n} \quad \textit{is an upper bound for} \quad S.$$

Problem 24. Prove this. *Hint:* Let $[b]$ be an upper bound for S, that is $[s] \leq [b]$ for all $[s] \in S$. Choose any $[s_0] \in S$. Use Lemma 23 to find a natural number N such that

$$N > n([b] - [s_0]).$$

Show that

$$k \geq N \quad \text{implies} \quad [s_0] + \frac{k}{n} > [s_0] + ([b] - [s_0]) = [b].$$

Therefore the set

$$\mathcal{K} = \left\{ k : [s_0] + \frac{k}{n} \text{ is } \textbf{not} \text{ an upper bound for } S \right\}$$

is a subset of $\{0, 1, 2, 3, \dots, N-1\}$ and thus is finite. Whence $\mathcal{K}$ has a largest element k_0. Let k_0 be the largest element of $\mathcal{K}$. Then $[s_0] + k_0/n$ is not an upper bound for S, and so there is an element $[s_n] \in S$ with

$$[s_0] + k_0/n < s_n.$$

But

$$[s_n] + \frac{1}{n} \geq [s_0] + \frac{k_0 + 1}{n}$$

and by the definition of k_0 as the largest element of $\mathcal{K}$ we find that $[s_0] + (k_0 + 1)/n$ is an upper bound for S. If follows that $[s_n] + 1/n$ is an upper bound for S. $\square$

Theorem 27. *The field $\mathbb{R}$ satisfies the least upper bound axiom. That is every nonempty $S \subseteq \mathbb{R}$ which is bounded above has a least upper bound.*

Problem 25. Prove this. *Hint:* Let S be a subset of $\mathbb{R}$ that is bounded above. By Lemma 26 for each natural number n there is a $[s_n] \in S$ such that $[s_n] + 1/n$ is an upper bound for S. By Lemma 22 there is a rational number r_n with

$$[s_n] < r_n < [s_n] + \frac{1}{n}.$$

The idea now is to show that $\langle r_1, r_2, r_3, \dots\rangle$ is a Cauchy sequence and therefore $[r] := [\langle r_1, r_2, r_3, \dots\rangle]$ is an element of $\mathbb{R}$ and that this element is the least upper bound of $\mathbb{R}$.

(a) Let N be a natural number and let $m, n \ge N$. Then $[s_n] \in S$ and $[s_m] + 1/m$ is an upper bound for S and thus

$$[s_n] \le [s_m] + \frac{1}{m} \le [s_m] + \frac{1}{N}$$

There is an similar inequality with m and n interchanged. Use this to show

$$m, n \ge N \quad \text{implies} \quad \big|[s_m] - [s_n]\big| \le \frac{1}{N}.$$

(b) From the definition of r_n we have

$$\big|[s_n] - r_n\big| < \frac{1}{n}.$$

Use the adding and subtracting trick to show

$$|r_m - r_n| \le \big|r_m - [s_m]\big| + \big|[s_m] - [s_n]\big| + \big|[s_n] - r_n\big|$$

and use this to show that for any natural number N

$$m, n \ge N \quad \text{implies} \quad |r_m - r_n| < \frac{3}{N}$$

(c) Show $\langle r\rangle = \langle r_1, r_2, r_3, \dots\rangle$ is a Cauchy sequence and thus $\langle r\rangle \in \mathcal{R}$. Therefore $[r] = [\langle r_1, r_2, r_3, \dots\rangle]$ is a well defined element of $\mathbb{R}$.

(d) For each n we have $[s_n] + 1/n$ is an upper bound for S, and

$$r_n + \frac{1}{n} > [s_n] + \frac{1}{n}$$

and therefore the rational number $r_n + 1/n$ is an upper bound for S. It follows that if $\langle z\rangle$ is defined to be

$$\langle z\rangle = \langle 1/n\rangle_{n=1}^{\infty} = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\rangle$$

so that

$$\langle r\rangle + \langle z\rangle = \left\langle r_1 + 1, r_2 + \frac{1}{2}, r_3 + \frac{1}{3}, r_4 + \frac{1}{4}, \dots\right\rangle$$

then every element in the sequence $\langle r\rangle + \langle z\rangle$ is an upper bound for S. Use this and Lemma 20 to show that $[r] + [z]$ is an upper bound for S.

(e) Use that $\lim_{n\to\infty} 1/n = 0$ to show $[z] = [0]$ and therefore

$$[r] = [r] + [0] = [r] + [z]$$

is an upper bound for S.

(f) Let $[b]$ be an upper bound for S. Then for each n there holds $[s_n] \le [b]$ because $[s_n] \in S$ and $[b]$ is an upper bound for S. We also have for each n that the rational number $r_n - 1/n$ satisfies

$$r_n - \frac{1}{n} < [s_n] \le [b].$$

Therefore each element in the sequence

$$\langle r\rangle - \langle z\rangle = \left\langle r_1 - 1, r_2 - \frac{1}{2}, r_3 - \frac{1}{3}, r_4 - \frac{1}{4}, \dots \right\rangle$$

is less than $[b]$. Use Lemma 25 to show

$$[r] = [r] - [z] \le [b].$$

(g) Put the pieces together and show that $[r]$ is a least upper bound for S. □

This finishes our construction of an ordered field that satisfies the least upper bound axiom.

2. Uniqueness of complete ordered fields.

Let $\mathbb{F}$ be an ordered field which where the least upper bound axiom holds. We wish to show that $\mathbb{F}$ ***isomorphic*** to the real numbers, $\mathbb{R}$. That is we wish to construct a map $f\colon \mathbb{R} \to \mathbb{F}$ such that f is bijective (that is one to one and onto) and such that for all $x, y \in \mathbb{R}$ that

$$f(x+y) = f(x) + f(y)$$
$$f(xy) = f(x)f(y)$$

and

$$x > 0 \quad \text{if and only if} \quad f(x) > 0.$$

Proposition 28. *Any ordered field $\mathbb{F}$ contains a copy of the rational numbers $\mathbb{Q}$. More precisely there is a function $\phi\colon \mathbb{Q} \to \mathbb{F}$ that is injective (that is one to one) such that for all $r, s \in \mathbb{Q}$*

$$\phi(r+s) = \phi(r) + \phi(s)$$
$$\phi(rs) = \phi(r)\phi(s)$$

Moreover the elements of $\mathbb{Q}$ that are positive in the usual sense are the elements of $\mathbb{Q}$ that are positive in $\mathbb{F}$, that is for $r \in \mathbb{Q}$,

$$r >_{\mathbb{Q}} 0 \quad \textit{if and only if} \quad \phi(r) >_{\mathbb{F}} 0.$$

Proof. We have $1 \in \mathbb{F}$ and we have seen in class that $1 > 0$ in $\mathbb{F}$ (for $1 = 1^2$ and squares are always positive). Let n be any positive integer. Then

$$n = \overbrace{1 + 1 + \cdots + 1}^{n \text{ terms}}$$

Thus we can view n as an element of $\mathbb{F}$ and n is positive in $\mathbb{F}$ as it is a sum of positive elements of $\mathbb{F}$. Let $r = a/b$ be a positive rational number □

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