Mathematics 554H/703I Final Name: Answer Key

1. Find the sum of the series $S = \sum_{k=1}^{10}(-1)^{k+1}(1-x)^k$.

Solution. This is a finite geometric series.

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{(-1)^{1+1}(1-x)^{1+1} - (-1)^{10+1}(1-x)^{10+1}}{1-(-(1-x))} \\ &= \frac{(1-x)^2 + (1-x)^{11}}{2-x} \end{aligned}$$

□

2. Give examples of

(a) A subset of $\mathbb{R}$ that is neither open or closed.

Solution. Maybe the most natural example is a half open interval $[a, b)$ (or $(a, b]$. Anther natural example is the set $\mathbb{Q}$ of rational numbers. □

(b) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at every point other than $x = 0$.

Solution. Here are several examples:

$$f(x) = \begin{cases} 0, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0; \\ 42, & x = 0. \end{cases}$$

$$f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$$

The most common way to have lost a bit of credit on this problem was to not have the function defined for $x = 0$. □

(c) A $f\colon \mathbb{R} \to \mathbb{R}$ such that f^2 is continuous, but f is not continuous. (*Hint:* $(-1)^2 = 1^2 = 1$.)

Solution. An easy example is

$$f(x) = \begin{cases} -1, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

which is not continuous at $x = 0$. Anther example is

$$f(x) \begin{cases} -1 & ,x \in \mathbb{Q}; \\ 1, & x \notin \mathbb{Q} \end{cases}$$

which is not continuous at any point. In both of these examples $f(x)^2$ is the constant function 1 which is continuous at all points. $\square$

(d) A compact subset of $\mathbb{R}$ that is not connected.

Solution. We need a closed bounded set that is not connected. The easiest example is a two element set such as $\{0, 1\}$. Anther example is the union of two disjoint closed intervals: $[0, 1] \cup [2, 3]$. $\square$

(e) A connected subset of $\mathbb{R}$ that is not compact.

Solution. The connected subsets of $\mathbb{R}$ are the intervals. So we are looking for an interval that is not closed and bounded. The open interval $(0, 1)$ works. As does the unbounded closed interval $[0, \infty)$. $\square$

3. Let $S \subseteq \mathbb{R}$ be a nonempty subset of the real numbers, $\mathbb{R}$.

(a) Define what it means for $b \in \mathbb{R}$ to be an ***upper bound*** for S.

Solution. This means that $s \leq b$ for all $s \in S$. $\square$

(b) Define what it means for β to be a ***least upper bound*** for S (denoted by $\beta = \sup(S)$).

Solution. That β is an upper bound for S and $\beta \leq b$ for all upper bounds b of S. $\square$

(c) State the ***Least Upper Bound Axiom***.

Solution. Every subset of $\mathbb{R}$ that is bounded above has a least upper bound. $\square$

(d) Use the Least Upper Bound Axiom to show that the set

$$S = \{1.001, (1.001)^2, (1.001)^3, (1.001)^4, \ldots\}$$

has no upper bound in $\mathbb{R}$.

Solution. Towards a contraction assume that S is bounded above. Then by the least Least Upper Bound Axiom the set S has a least upper bound $\beta = \sup(S)$. Let n be any natural number. Then $(1.001)^{n+1} \in S$ and β is an upper bound for S and thus

$$(1.001)^{n+1} \leq \beta.$$

Dividing by (1.001) gives

$$(1.001)^n \leq \frac{\beta}{1.001} < \beta$$

for all natural numbers n. This implies that $\beta/(1.001)$ is an upper bound for S, contradicting that β is the *least* upper bound for S. $\square$

4. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in the metric space E.

(a) Define what it means for $\lim_{n\to\infty} p_n = p$.

Solution. For every $\varepsilon > 0$, there is a N such that $n \geq N$ implies $d(p, p_n) < \varepsilon$. $\square$

(b) Define what if means for $\langle p_n \rangle_{n=1}^{\infty}$ to be a ***Cauchy sequence***.

Solution. For every $\varepsilon > 0$ there is a N such that $m, n \geq N$ implies $d(p_m, p_n) < \varepsilon$. $\square$

(c) Show that if $\langle p_n \rangle_{n=1}^{\infty}$ converges, then it is a Cauchy sequence.

Solution. Let $\varepsilon > 0$. As the sequence converges, it converges to some $p \in E$, that is $\lim_{n\to\infty} p_n = p$. This implies there is a N such that

$$n \geq N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

Thus if $m, n \geq N$ we have that both the inequalities

$$d(p_m, p), d(p_n, p) < \frac{\varepsilon}{2}$$

hold. Thus if $m, n \geq N$ we have, by the triangle inequality,

$$d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus the sequence is Cauchy. $\square$

5. Let E be a metric space and $U \subseteq E$ a subset of E.

(a) Define what it means for U to be an ***open set***.

Solution. For every $p \in U$ there is a $r > 0$ such that $B(p, r) \subseteq U$. (Here $B(p, r) = \{x \in E : d(x, r) < r\}$ is the open ball of radius r about p in E.) $\square$

(b) Let $\{U_\alpha\}_{\alpha \in I}$ be a possibly infinite collection of subsets of E define the union $U = \bigcup_{\alpha \in I} U_\alpha$.

Solution. The union is

$$U = \{x : x \in U_\alpha \text{ for at least one } \alpha \in I\}.$$

$\square$

(c) In part (b) show that if each U_α is open, then the union, U, is also open.

Solution. Let $p \in U$. We need to find a $r > 0$ so that $B(p,r) \subseteq U$. By the definition of union we have $p \in U_\alpha$ for at least one α. As U_α is open there is a $r > 0$ such that $B(p,r) \subseteq U_\alpha$. But $U_\alpha \subseteq U$ and thus

$$B(p,r) \subseteq U_\alpha \subseteq U.$$

and so $B(p,r) \subseteq U$. Therefore U is open. $\square$

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + 1$. Prove directly from the ε–δ definition of limit that $\lim_{x\to 2} f(x) = 9$.

Solution. We first do a preliminary calculation:

$$\begin{aligned} |f(x) - f(2)| &= |(x^3+1) - (2^3 - 1)| \\ &= |x^3 - 2^2| \\ &= |x-2|\big(|x^2 + 2x + 4|\big) \\ &\le |x-2|\big(|x|^2 + 2|x| + 4|\big) \end{aligned}$$

Now assume that $|x - 2| < 1$. Then

$$|x| = |2 + (x-2)| \le 2 + |x-2| < 2 + 1 = 3.$$

Thus when $|x-2| < 1$ we have

$$|x|^2 + 2|x| + 4 \le 3^2 + 2(3) + 4 = 19.$$

Using this in the preliminary calculation show that if $|x-2| < 1$, then

$$|f(x) - f(2)| < 19|x-2|.$$

Let $\varepsilon > 0$ and set

$$\delta = \min\{1, \frac{\varepsilon}{19}\}.$$

Then if $|x-2| < \delta$ we have $|x-2| < 1$ and so

$$\begin{aligned} |f(x) - f(2)| &\le 19|x-2| \\ &< 19\delta \\ &\le 19\frac{\varepsilon}{19} \\ &= \varepsilon. \end{aligned}$$

That is $|x-2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$. This is exactly the definition of $\lim_{x\to 2} f(x) = f(2)$. $\square$

7. Let E be a metric space.

(a) Define what it means for E to be ***complete***.

Solution. That every Cauchy sequence in E converges to a point of E. ☐

(b) Define what it means for E to be ***sequentially compact***.

Solution. Every sequence in E has a subsequence that converges to a point of E. ☐

(c) Prove that any sequentially compact space is complete.

Solution. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. As E is sequentially compact there is a subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ that converges to some point p of E. (That is $\lim_{k\to\infty} p_{n_k} = p$.) But we have shown that if a Cauchy sequence has a convergent subsequence, that the original sequence also converges and converges to the same limit as the subsequence. Therefore $\lim_{n\to\infty} p_n = \lim_{k\to\infty} p_{n_k} = p$. Thus we have shown that any Cauchy sequence in E converges to a point of E and therefore E is complete.

The proof here is pretty much what I expected, but some of you went the extra mile and proved the result about Cauchy sequences with a convergent subsequence also converging. While this did not get you any extra points, it did impress me. ☐

8. (a) State the ***intermediate value theorem*** for functions $f\colon [a,b] \to \mathbb{R}$.

Solution. Let $f\colon [a,b]$ be a continuous function and assume that $f(a)$ and $f(b)$ have opposite signs. (This is one is positive and the other is negative.) Then there is a points $\xi(a,b)$ with $f(\xi) = 0$; ☐

(b) Use the intermediate value theorem to prove that every positive real number has a fourth root. (You may assume polynomials are continuous.)

Solution. Let $c > 0$ and let f be the polynomial

$$f(x) = x^4 - c.$$

Any solution of $f(x) = 0$ will be a fourth root of c. So it is enough to show that $f(x) = 0$ has a root. Note that

$$f(0) = -c < 0.$$

And

$$f(1+c) = (1+c)^4 - c = 1+4c+6c^2+4c^3+c^4-c = 1+3c+6c^2+4c^3+c^4 > 0$$

Thus $f(x)$ is continuous on $[0, 1+c]$ and $f(0)$ and $f(1+c)$ have opposite signs. Therefore, by the Intermediate Value Theorem there is a $\xi \in (0, 1+c)$ with $f(\xi) = 0$. This ξ is fourth root of x. □

9. Let E be a metric space and $p, q \in E$ and let

$$S = \{x : d(x,p) + d(x,q) = 1\}$$

Show that S is a closed subset of E.

This was the least popular problem on the exam. So I am including two solutions.

Solution using preimages of closed sets. We know that the preimages of closed sets by continuous functions are closed.

We have seen many times that the functions

$$f_p(x) = d(x,p)$$
$$f_q(x) = d(x,q)$$

are continuous. Thus the sum

$$f(x) = f_p(x) + f_q(x) = d(x,p) + d(x,q)$$

is continuous. The set S is

$$S = \{x : f(x) = 1\} = f^{-1}[\{1\}].$$

The one element set $\{1\}$ is closed and therefore S is the preimage of a closed set by a continuous functions and thus S is closed. □

Solution using limits. We know that a set that contains all the limits of its convergent sequences is closed.

Let $\langle x_n \rangle_{n=1}\infty$ be a sequence in S and assume that $\lim_{n\to\infty} x_n = x$. To complete the proof we need to show that $x \in S$. As for each n we have $p_n \in S$ we have

$$d(x_n,p) + d(x_n,q) = 1$$

Taking the limit is this and using that the distance form a point is continuous we have

$$\begin{aligned} d(x,p) + d(x,q) &= \lim_{n\to\infty} \big(d(x_n,p) + d(x_n,q)\big) \\ &= \lim_{n\to\infty} 1 \\ &= 1. \end{aligned}$$

Thus $x \in S$ and we are done. □

10. Let E be a metric space and $f\colon E \to \mathbb{R}$ be a function. Recall that f is ***bounded*** if and only if there is a constant $M > 0$ such that

$$|f(x)| \le M \quad \text{for all} \quad x \in E.$$

The function f is ***locally bounded*** if and only if for every $p \in E$ there is a $r_p > 0$ and a constant $M_x > 0$ such that

$$|f(x)| \le M_p \quad \text{for all} \quad x \in B(p, r_p).$$

Prove that on a compact metric space that a locally bounded function is bounded. *Hint:* One way to start is to note that $\{B(p, r_p) : p \in E\}$ is an open cover of E.

Solution. As per the hint let $\mathcal{U} = \{B(p, r_p) : p \in E\}$. Then each element of $\mathcal{U}$ is an open ball and thus open. And if $p \in E$, then $p \in B(p, r_p) \in \mathcal{U}$. Thus $\mathcal{U}$ is a cover of E. As E is compact this open cover has a finite subcover. Thus there are a finite set of points $p_1, p_2, \ldots, p_n$ such that

$$E \subseteq \bigcup_{k=1}^{n} B(p_k, r_{p_k}).$$

Let M_{p_k} be the constant such that $|f(x)| \le M_{p_k}$ for all $x \in B(p_k, r_{p_k})$. Set

$$M = \max\{M_{p_1}, M_{p_2}, \ldots, M_{p_n}\}.$$

(This maximum exists as the set is finite.) For any $x \in E$ we have that $x \in B(x_k, r_{p_k})$ for at least one k with $1 \le k \le n$. Thus

$$|f(x)| \le M_{p_k} \le M.$$

As x was any arbitrary point of E this shows that f is bounded on E. □