Mathematics 554H/703I Test 1 Name: AnswerKey.

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1. What is the sum of the series $S = \sum_{k=1}^{100} \frac{(-1)^k}{x^k}$?

Solution. This is a geometric series.

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{\frac{-1}{x} - \frac{(-1)^{101}}{x^{101}}}{1 - \frac{-1}{x}} \qquad \text{(ok to have stopped here)} \\ &= \frac{-x^{100} + 1}{x^{101} + x^{100}} \\ &= \frac{1 - x^{100}}{x^{101} + x^{100}} \end{aligned}$$

□

2. (a) Define the binomial coefficient $\binom{n}{k}$

Solution. The definition is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n and k are nonnegative integers and $0 \le k \le n$. □

(b) Simplify $\frac{(x+h)^3 - (x-h)^3}{h}$ (the answer should have no h in the denominator).

Solution. Use the binomial theorem to expand the binomials.

$$\begin{aligned} (x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3 \\ (x-h)^3 &= x^3 - 3x^2h + 3xh^2 - h^3 \end{aligned}$$

Subtracting these gives

$$(x+h)^3 - (x-h)^3 = 6x^2h + 2h^3 = h(6x^2 + h^2)$$

and therefore

$$\frac{(x+h)^3 - (x-h)^3}{h} = 6x^2 + 2h^3$$

3. For $\mathbf{a} = (a_1, a_n, \dots, a_n)$ and $\mathbf{b} \in (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ define the following:

(a) $\mathbf{a}\cdot\mathbf{b}$. **Solution.** $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.
(b) $\|\mathbf{a}\|$. **Solution.** $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

State the following:
(c) The ***Cauchy-Schwartz*** inequality for vectors in $\mathbb{R}^n$. **Solution.** $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$
(d) The ***triangle inequality*** for vectors in $\mathbb{R}^n$. **Solution.** $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$
4. Give an example of a function $f\colon \mathbb{R}\to\mathbb{R}$ and subsets $A, B \subseteq \mathbb{R}$ such that $f(A\cap B) \ne f(A)\cap f(B)$.

Solution. There are many such examples. An easy one is let $f\colon \mathbb{R}\to\mathbb{R}$ be $f(x) = x^2$ and let A and B be the one element sets $A = \{-1\}$ and $B = \{1\}$. Then $A\cap B = \varnothing$, but $f[A] = f[B] = \{1\}$. Thus

$$f[A\cap B] = \varnothing \ne \{1\} = f[A]\cap f[B].$$

□

5. Give an example of a subset of $\mathbb{R}$ which is bounded below, but which does not have a minimum.

Solution. And easy example here is the open interval $(0,1)$. This is bounded below (by 0). But there is no minimum. For if $x\in(0,1)$, then also $x/2 \in (0,1)$ and $x/2 < x$, thus no $x\in(0,1)$ can be a minimum of the set. □

6. Let $f\colon X\to Y$ be a function between the two sets X and Y.
(a) If $U\subseteq Y$ define $f^{-1}[U]$.

Solution. The definition is

$$f^{-1}[U] = \{x\in X : f(x)\in U\}.$$

□

(b) If $\{U_\alpha : \alpha\in I\}$ is a collection of subsets of Y define $\bigcap_{\alpha\in I} U_\alpha$.

(c) The definition is

$$\bigcap_{\alpha\in I} U_\alpha = \{x : x\in U_\alpha \text{ for all } \alpha\in I\}.$$

□

(d) Prove $f^{-1}\Big[\bigcap_{\alpha\in I} U_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[U_\alpha]$.

Solution.

$$\begin{aligned} x \in f^{-1}\Big[\bigcap_{\alpha\in I} U_\alpha\Big] &\iff f(x) \in \bigcap_{\alpha\in I} U_\alpha \\ &\iff \text{for all } \alpha \in I, \quad f(x) \in U_\alpha \\ &\iff \text{for all } \alpha \in I, \quad x \in f^{-1}[U_\alpha] \\ &\iff \bigcap_{\alpha\in A} f^{-1}[U_\alpha]. \end{aligned}$$

□

7. Let $f\colon [0,2] \to \mathbb{R}$ be the function

$$f(x) = 2x^3 - 2x^2 - 1.$$

(a) Prove there is a constant M such that

$$|f(x) - f(y)| \le M|x - y|$$

for all $x, y \in [0,2]$.

Solution. Let $x, y \in [0,2]$

$$\begin{aligned} |f(x) - f(y)| &= |2x^3 - 2x^2 - 1 - (2y^3 - 2y^2 - 1)| \\ &= |2(x^3 - y^3) - 2(x^2 - ^2)| \\ &= 2|x-y||(x^2 + xy + y^2) - (x+y)| \\ &\le 2|x-y|\big(|x|^2 + |x||y| + |y|^2 + |x| + |y|\big) && \text{(triangle inequality)} \\ &\le 2|x-y|\big(2^2 + 2\cdot 2 + 2^2 + 2 + 2\big) && \text{(as } |x|, |y| \le 2) \\ &= 32|x-y|. \end{aligned}$$

Therefore $M = 32$ works. □

(b) Prove that there is a point $\xi \in (0,2)$ with $f(\xi) = 0$.

Solution. The form of the Intermediate Value Theorem we have proven is that if $f\colon [a,b] \to \mathbb{R}$ satisfies a Lipschitz condition and also $f(a) < 0$ and $f(b) > 0$ then there is $\xi \in (a,b)$ with $f(\xi) = 0$. Part (b) of this problem shows that $f(x)$ is Lipschitz on the interval $[a,b] = [0,2]$. And

$$\begin{aligned} f(0) &= 2(0)^3 - 2(0)^2 - 1 = -1 < 0 \\ f(2) &= 2(2)^2 - 2(2)^2 - 1 = 7 > 0 \end{aligned}$$

so the Intermediate Value Theorem applies and thus there is $\xi \in (0,2)$ with $f(\xi) = 0$. □

8. Let $a > 0$ and let x be so that $|x - a| < \dfrac{a}{2}$ and $|x - a| < \delta$.

(a) Show $\dfrac{a}{2} \le x \le \dfrac{3a}{2}$.

Solution. We first show the lower bound:

$$\begin{aligned} x &= a + (x-a) \\ &\geq a - |x-a| && \text{(as } (x-a) \geq -|x-a|) \\ &\geq a - \frac{a}{2} && \text{(as } -|x-a| \geq -a/2) \\ &= \frac{a}{2}. \end{aligned}$$

And now the upper bound:

$$\begin{aligned} x &= a + (x-a) \\ &\leq a + |x-a| && \text{(as } (x-a) \leq |x-a|) \\ &\leq a + \frac{a}{2} && \text{(as } |x-a| \geq a/2) \\ &= \frac{3a}{2}. \end{aligned}$$

□

Solution 2. We are given

$$|x-a| \leq \frac{a}{2}.$$

This implies

$$-\frac{a}{2} \leq x - a \leq \frac{a}{2}.$$

Add a to this inequality to get

$$\frac{a}{2} \leq x \leq \frac{3a}{2}.$$

(b) Show $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \frac{10\delta}{a^3}$.

Solution.

$$\begin{aligned}
\left|\frac{1}{x^2}-\frac{1}{a^2}\right| &= \frac{a^2-x^2}{x^2a^2}\\
&= \frac{|a+x||a-x|}{x^2a^2a}\\
&< \frac{|a+x|\delta}{x^2a^2a} && \text{(as } |x-a|<\delta)\\
&\le \frac{(a+x)\delta}{x^2a^2}\\
&\le \frac{(a+3a/2)\delta}{(a/2)^2a^2} && \text{(as } x\le 3a/2 \text{ and } 1/x\le 1/(a/2))\\
&= \frac{10\delta}{a^3}
\end{aligned}$$

□

9. (a) Let $S \subseteq \mathbb{R}$ be a nonempty subset of $\mathbb{R}$. Define what it means for S to be ***bounded above.***

Solution. The set S is bounded above if there is a $c \in \mathbb{R}$ such that $s \le c$ for all $s \in S$. □

(b) Define what it means for b to be a least upper bound of S.

Solution. The number b is a least upper bound if b is an upper bound for S and $b \le c$ for all upper bounds c. □

(c) State the ***least upper bound axiom***.

Solution. Every nonempty subset of $\mathbb{R}$ that has a upper bound, has a least upper bound. □

(d) State ***Archimedes' axiom***.

Solution. For any real number, x, there is a natural number n such that $n > x$. □

(e) Use the least upper bound axiom to prove Archimedes's axiom (big form).

Solution. Towards a contradiction, assume this is false. Then there is an $x \in \mathbb{R}$ such that for all natural numbers n the inequality $n \le x$ holds. This implies that the set, $\mathbb{N}$, of natural numbers has an upper bound. Let $b = \sup(\mathbb{N})$ be the least upper bound for $\mathbb{N}$. Then for any natural number m we have

$$m \le b.$$

But for any natural number n the number $m = n + 1$ is a natural number and whence

$$n + 1 \le b.$$

This implies that for all $n \in \mathbb{N}$ that

$$n \leq b - 1$$

and thus $b - 1$ is a upper bound for $\mathbb{N}$. But $b - 1 < b$, contradicting that b was the least upper bound. $\square$

10. (a) Define the ***open ball***, $B(a, r)$, with center a and radius r in the metric space E.

Solution. $B(a, r) = \{x \in E : d(a, x) < r\}$.

(b) Define what it means for the set U to be ***open*** in the metric space E.

Solution. The set $U \subseteq E$ is open if and only if for all points $a \in U$ there is a $r > 0$ such that $B(a, r) \subseteq U$. $\square$

(c) Let U and V be open sets in E. Prove $U \cap V$ is also open.

Solution. Let $a \in U \cap V$. Then $a \in U$ and $a \in V$. As U is open there is $r_1 > 0$ such that $B(a, r_1) \subseteq U$. As V is open there is $r_2 > 0$ such that $B(a, r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then $r > 0$ and $r \leq r_1$ and $r \leq r_2$. Thus

$$B(a, r) \subseteq B(a, r_1) \subseteq U \quad \text{and} \quad B(a, r) \subseteq B(a, r_2) \subseteq V.$$

This implies

$$B(a, r) \subseteq U \cap V.$$

As a was any point of $U \cap V$ this implies $U \cap V$ is open. $\square$