Mathematics 554H/703I Test 2 Name: **Answer key**

1. Define the following:

(a) The point p is an ***adherent point*** of the set S.

Solution. p is an adherent point of S if and only if for all $r > 0$ $S \cap B(p, r) \neq \varnothing$. $\square$

(b) $\lim_{n\to\infty} p_n = p$ in the metric space E.

Solution. For every $\varepsilon > 0$ there is a N such that if $n \geq N$, then $d(p_n, p) < \varepsilon$. $\square$

(c) $\mathcal{U}$ is an ***open cover*** of S.

Solution. $\mathcal{U}$ is an open cover of $S \subseteq E$ where E is a metric space if and only if each $U \in \mathcal{U}$ is an open set and $S \subseteq \bigcup_{U\in\mathcal{U}} U$. $\square$

(d) The set S is ***sequentially compact***.

Solution. This means that if $\langle p_n\rangle_{n=1}^{\infty}$ is a sequence in S, that there is a subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$ that converges to some point of S. $\square$

(e) The set S is ***compact***.

Solution. Every open cover $\mathcal{U}$ of S has a finite subset $\mathcal{U}_0$ with $S \subseteq \bigcup_{U\in\mathcal{U}_0}$. (Or more briefly, every open cover of S has a finite sub cover. $\square$

2. State the following:

(a) The ***BolzanoWeierstrass theorem***.

Solution. Every closed bounded subset of $\mathbb{R}^n$ is sequentially compact.

Also accepted: Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. $\square$

(b) The ***Lebesgue Covering Lemma***.

Solution. Let E be a compact metric space and $\mathcal{U}$ an open cover of E. Then there is an $r > 0$ such that for all $x \in E$ there is a $U\mathcal{U}$ with $B(x, r) \subseteq U$. $\square$

3. Let $\langle x_n\rangle_{n=1}^{\infty}$ be an increasing sequence of real numbers that is bounded above. Use the least upper bound property of the real numbers to prove the sequence converges.

Solution. As the sequence is bounded above the set $\{a_n : n = 1, 2, 3, \ldots\}$ is bounded above and therefore by the least upper bound axiom this set has a least upper bound:

$$\alpha = \sup\{a_n : n = 1, 2, 3, \ldots\}$$

exists. Let $\varepsilon > 0$. Then as α is the least upper bound for for the set that is N such that

$$\alpha - \varepsilon < a_N$$

otherwise α would not be the smallest upper bound. As the sequence is increasing if $n \geq N$ we havd

$$\alpha - \varepsilon < a_N \leq a_n \leq \alpha.$$

That is if $n \geq N$

$$-\varepsilon < a_n - \alpha \leq 0.$$

This implies that $n \geq N$ implies that $|a_n - \alpha| < \varepsilon$ and so $\lim_{n\to\infty} a_n = \alpha$ and thus the sequence converges. □

4. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers with $\lim\limits_{n\to\infty} x_n = L$. Give a N, ε proof that $\lim\limits_{n\to\infty} x_n^2 = L^2$.

Solution. As a preliminary computation note

$$|x_n^2 - L^2| = |x_n + L||x_n - L| \leq (|x_n| + |L|)|x_n - L|.$$

As $\lim_{n\to\infty} x_n = L$, there is a N_1 such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - L| < \varepsilon.$$

When this holds we have

$$|x_n| = |L + (x_n - L)| \leq |L| + |x_n - L| < |L| + 1.$$

Using this in our preliminary calculation gives that for $n \geq N_1$

$$|x^2 - L^2| \leq (|x_n| + |L|)|x_n - L| \leq (|L| + 1 + |L|)|x_n - L| = (2|L| - 1)|x_n - L|.$$

Let $\varepsilon > 0$, then there is a N_2 such that

$$n \geq N_2 \quad \text{implies} \quad |x_n - L| \leq \frac{\varepsilon}{2|L| + 1}$$

Let $N = \max\{N_1, N_2\}$. Then putting our inequalities together gives that for $n \geq N$ we have

$$\begin{aligned} |x_n^2 - L^2| &\leq (2|L| + 1)|x_n - L| \\ &< (2|L| + 1)\frac{\varepsilon}{2|L| + 1} \\ &= \varepsilon. \end{aligned}$$

Therefore $\lim_{n\to\infty} x_n^2 = L^2$. □

5. Let E be a metric space and $f\colon E \to \mathbb{R}$ a function that satisfies

$$|f(p) - f(q)| \leq 2d(p, q).$$

Show the set $U = \{x \in E : f(x) < 10\}$ is open.

Solution. Let $x_0 \in U$. Then $f(x_0) < 10$. Set

$$r = \frac{10 - f(x_0)}{2}.$$

This is positive as $f(x_0) < 10$. Let $x \in B(x_0, r)$. Then $d(x, x_0) < r$ and therefore

$$\begin{aligned} f(x) &= f(x_0) + (f(x) - f(x_0)) \\ &\leq f(x_0) + |f(x) - f(x_0)| \\ &\leq f(x_0) + 2d(x, x_0) \\ &< f(x_0) + 2r \\ &= f(x_0) + 2\frac{10 - f(x_0)}{2} \\ &= 10. \end{aligned}$$

That is if $x \in B(x_0, r)$, then $f(x) < 0$. Thus $B(x_0, r) \subseteq U$. This shows that any $x_0 \in U$ is the center of some ball contained in U and thus U is open. □

6. Let $x_1, x_2, x_3, \ldots$ be real numbers defined by

$$\begin{aligned} x_1 &= 17 \\ x_{n+1} &= 42 + \frac{x_n}{3}. \end{aligned}$$

Show that $\langle x_n \rangle_{n=1}^{\infty}$ converges and find its limit.

Solution 1. Let

$$f(x) = 42 + \frac{x}{3}.$$

Then $x_{n+1} = f(x_n)$. If the limit exists, say that

$$\lim_{n\to\infty} x_n = \xi$$

then taking the limit in $x_{n+1} = x_n$ and gives

$$\xi = 42 + \frac{\xi}{3}$$

which can be solved for ξ to get

$$\xi = 63.$$

So we make the guess that 63 is the limit. Also note that for all $x, y \in \mathbb{R}$ that

$$|f(x) - f(y)| = \left|42 + \frac{x}{3} - \left(42 - \frac{y}{3}\right)\right| = \frac{1}{3}|x - y|.$$

Using that $f(63) = 63$ we have for any $x \in R$

$$|f(x) - 63| = |f(x) - f(63)| = \frac{1}{3}|x - 63|.$$

Thus for each $n \geq 0$

$$|x_{n+1} - 63| = |f(x_n) - 63| = \frac{1}{3}|x_n - 63|$$

which implies

$$\begin{aligned}
|x_1 - 63| &= \frac{1}{3}|x_0 - 63| = \frac{17 - 63}{3} = \frac{46}{3} \\
|x_2 - 63| &= \frac{1}{3}|x_1 - 63| = \frac{46}{3^2} \\
|x_3 - 63| &= \frac{1}{3}|x_2 - 63| = \frac{46}{3^3}
\end{aligned}$$

and after n steps

$$|x_n - 63| = \frac{46}{3^n}$$

So $\varepsilon > 0$ there is a positive integer N with

$$\frac{46}{3^N} < \varepsilon.$$

Whence if $n \geq N$ we have

$$|x_n - 63| = \frac{46}{3^n} \leq \frac{46}{3^N} < \varepsilon$$

and therefore $\lim_{n\to\infty} x_n = 63$. □

Solution 2. First note

$$x_1 = 42 + \frac{x_0}{3} = 42 + \frac{17}{3} > 17 = x_0.$$

Therefore $x_1 > x_0$. Assume that $x_{n+1} > x_n$. Then

$$x_{n+2} = 42 + \frac{x_{n+1}}{3} > 43 + \frac{x_n}{3} = x_{n+1}.$$

It then follows by induction that the sequence is increasing.

Also note that $x_0 < 63$. Assume that $x_n < 63$, then

$$x_{n+1} = 42 + \frac{x_n}{3} < 42 + \frac{63}{3} = 63.$$

Thus anther induction implies that $x_n < 63$. Thus the sequence is an increasing sequence which is bounded above. Therefore it converges. So let

$$\xi = \lim_{n\to\infty} x_n$$

then

$$\xi = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(42 + \frac{\xi}{3}\right) = 42 + \frac{\xi}{3}.$$

Solve this for ξ gives

$$\xi = \lim_{n\to\infty} x_n = 63.$$

□

7. Let S be a nonempty subset of the metric space E and let q be an adherent point of S. Prove there is a sequence of points $\langle p_n\rangle_{n=1}^{\infty} \subseteq S$ with $\lim\limits_{n\to\infty} p_n = q$.

Solution. Let n be a positive integer. By the defintion of an adherent points the $S \cap B(q, 1/n) \neq \varnothing$. Thus we can choose a point $p_n \in S \cap B(q, 1/n)$. Then $p_n \in S$ and $d(p_n, q) < 1/n$. Let $\varepsilon > 0$. Then by Archimedes's axiom there is a positive integer N with $1/N < \varepsilon$. Then for $n \geq N$ we have

$$d(p_n, q) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $\langle p_n\rangle_{n=1}^{\infty}$ is a sequence of points from S with $\lim_{n\to\infty} p_n = q$. □

8. (a) Define what it means for a metric space to be ***complete***.

Solution. The metric space E is complete if and only if every Cauchy sequence in E converges to a point of E. □

(b) Let E be a complete metric space. Show that any nonempty closed subset of E is also a complete metric space. (You may assume that a closed set contains all the limits of all its convergent sequences.)

Solution. Let S be a closed subset of E and $\langle p_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in S. Then the sequence is also a Cauchy sequence in E and E is complete. Therefore there is a point $p \in E$ with $\lim_{n\to\infty} p_n = p$. But then p is a limit of a sequence of points from S and S is closed and therefore S contains the limits of its sequences. Thus $p \in S$. This shows that every Cauchy sequence in S converges to a point of S and thus S is complete. □