Mathematics 554H/703I Test 3 Name: **Answer Key**

1. (a) Let $f\colon E \to E'$ be a map between metric spaces and let $x_0 \in E$. Define what it means for f to be continuous at x_0. (The ε–δ definition.)

Solution. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for $x \in E$, we have $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \varepsilon$. □

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 2x^2$. Prove directly from the definition that f is continuous at the point $x = 2$.

Solution. We do a standard preliminary calculation:

$$|f(x) - f(2)| = |2x^2 - 2(2)^2| = 2|x+2||x-2| \le (|x| + 2)|x-2|.$$

If $|x - 2| < 1$, then we have

$$|x| = |2 + (x-2)| \le 2 + |x-2| < 2 + 1 = 3.$$

Using this in the preliminary calculation gives that

$$|f(x) - f(2)| \le (|x| + 2)|x - 2| \le (2+2)|x-2| = 4|x-2|$$

holds whenever $|x - 2| < 1$. Let $\varepsilon > 0$ and set

$$\delta = \min\left\{1, \frac{\varepsilon}{4}\right\}.$$

Then if $|x - 2| < \delta$ we havd

$$\begin{aligned} |f(x) - f(2)| &\le 4|x-2| \\ &< 4\delta \\ &\le 4\frac{\varepsilon}{4} \\ &= \varepsilon \end{aligned}$$

which shows that $f(x)$ is continuous at $x = 2$. □

(c) Let E be a metric space and $g, h\colon E \to \mathbb{R}$ functions that are continuous at the point $x_0 \in E$. Prove directly from the definition of continuity that $f = 3g - 4h$ is continuous at x_0.

Solution. Here we do a anther preliminary calculation:

$$\begin{aligned} |f(x) - f(x_0)| &= |(3g(x) - 4h(x)) - (3g(x_0) - 4h(x_0))| \\ &\le 3|g(x) - g(x_0)| + 4|h(x) - h(x_0)|. \end{aligned}$$

Let $\varepsilon > 0$. Then by the continuity of g at x_0, there is a $\delta_1 > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad |g(x) - g(x_0)| < \frac{\varepsilon}{6}.$$

The continuity of h at x_0 gives a $\delta_2 > 0$ such that

$$d(x, x_0) < \delta_2 \quad \text{implies} \quad |h(x) - h(x_0)| < \frac{\varepsilon}{8}.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then $d(x, x_0) < \delta$ implies

$$\begin{aligned} |f(x) - f(x_0)| &\le 3|g(x) - g(x_0)| + 4|g(x) - g(x_0)| \\ &< 3\frac{\varepsilon}{6} + 4\frac{\varepsilon}{8} \\ &= \varepsilon. \end{aligned}$$

This shows that f is continuous at x_0. □

2. Given example of (you do not have to prove your example works and defining a function by drawing its graph is acceptable)

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at every point other than $x = 2$ and $x = 3$.

Solution. One example would be

$$f(x) = \begin{cases} 1, & 2 \le x \le 3; \\ 1, & \text{otherwise}. \end{cases}$$

Anther example is

$$f(x) = \begin{cases} \dfrac{1}{(x-2)(x-3)}, & x \ne 2, 3 \\ 42, & x = 2 \text{ or } x = 3. \end{cases}$$

□

(b) A function $g\colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0} f(x)$ exists, but g is not continuous at 0.

Solution. Maybe the easiest example is

$$g(x) = \begin{cases} 0, & x \ne 0; \\ 1, & x = 0. \end{cases}$$

□

(c) A function $h\colon \mathbb{R} \to \mathbb{R}$ that is not continuous at any point of $\mathbb{R}$.

Solution. As we did in class the standard example is

$$h(x) = \begin{cases} 1, & x \text{ is a rational number}. \\ 0, & x \text{ is irrational}. \end{cases}$$

□

3. Let $f\colon E \to E'$ a map between metric spaces that is continuous at every point. Prove that if $U \subseteq E'$ is an open set in E', that the preimage $f^{-1}[U]$ is open in E. (That is prove that the preimage of open sets by continuous functions are open.)

Solution. Let $x_0 \in f^{-1}[U]$. We need to find a ball centered at x_0 that is contained in $f^{-1}[U]$. As $x_0 \in f^{-1}[U]$ we have $f(x_0) \in U$. As U is open there is a $\varepsilon > 0$ such that $B(f(x_0), \varepsilon) \subseteq U$. Because f is continuous at x_0 there is a $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad d(f(x), f(x_0)) < \varepsilon.$$

That is $d(x, x_0) < \delta$ implies

$$f(x) \in B(f(x_0), \varepsilon) \subseteq U.$$

This in turn implies $x \in f^{-1}[U]$. That is we have shown $x \in B(x_0, \delta)$ implies $x \in f^{-1}[U]$. Thus $B(x_0, \delta) \subseteq f^{-1}[U]$. Therefore $f^{-1}[U]$ is open. □

4. (a) Let E be a metric space. Define what it means for E to be ***connected***.

Solution. The metric space E is connected if and only if it is not the disjoint union of two nonempty open set. Explicitly this means that there are no nonempty open set A and B of E such that

$$E = A \cup B \quad \text{and} \quad A \cap B = \varnothing.$$

□

(b) Let E be a connected metric space and let $f\colon E \to \mathbb{R}$ be a continuous function. Let $p, q \in \mathbb{R}$ with $f(p) < 0$ and $f(q) > 0$. Prove there is an $x \in E$ with $f(x) = 0$.

Solution. Recall that if a metric space is not connected, then it has a ***disconnection***, that is disjoint nonempty open set A and B such that $E = A \cup B$.

Towards a contradiction assume that there is no $x \in E$ with $f(x) = 0$. Then the image $f[E] \subseteq (-\infty, 0) \cup (0, \infty)$ and therefore

$$E = f^{-1}[(-\infty, 0)] \cup f^{-1}[(0, \infty)].$$

As f is continuous and the sets $(-\infty, 0)$ and $(0, \infty)$ are both open the sets $f^{-1}[(-\infty, 0)]$ and $f^{-1}[(0, \infty)]$ are open. Both are nonempty as $p \in f^{-1}[(-\infty, 0)]$ and $q \in f^{-1}[(0, \infty)]$. Thus

$$E = f^{-1}[(-\infty, 0)] \cup f^{-1}[(0, \infty)]$$

is a disconnection of E, contradicting that E is connected. □

5. Prove that the equation $x^3 - 8x + 2 = 0$ has at least 3 real roots.

Solution. Let $f(x) = x^3 - 8x + 2$. Then f is a polynomial and thus is continuous. It is easy to check that

$$\begin{aligned} f(-3) &= -1 \\ f(0) &= 2 \\ f(1) &= -5 \\ f(3) &= 5. \end{aligned}$$

Then $f(x)$ has opposite signs on the endpoints of the interval $[-3, 0]$ and so there is a $x_1 \in (-3, 0)$ with $f(x_1) = 0$. Likewise on each of the intervals $[0, 1]$ and $[1, 3]$ $f(x)$ has opposite signs at the endpoints and whence that are $x_2 \in (0, 1)$ and $x_3 \in (1, 3)$ with $f(x_2) = f(x_3) = 0$. Therefore $f(x) = 0$ has at least three roots. □

6. Let E be a metric space and for $p \in E$ we have seen that the function $f_p \colon E \to \mathbb{R}$ given by

$$f_p(x) = d(x, p)$$

is continuous. Use this to show that if $p, q \in E$ that the set

$$S = \{x : d(x, p) > d(x, q)\}$$

is open.

Solution. Note that we can write S as

$$\begin{aligned} S &= \{x : d(x, p) > d(x, q)\} \\ &= \{x : f_p(x) > f_q(x)\} \\ &= \{x : f_p(x) - f_q(x) > 0\} \\ &= \{x : f(x) > 0\} \\ &= f^{-1}[(0, \infty)] \end{aligned}$$

where f is the functions

$$f(x) = f_p(x) - f_q(x).$$

The function f is the difference to two continuous functions and therefore f is continuous. This shows that S is the preimage of an open set by a continuous function and therefore S is open. □