Math 172 Homework.

We have been about discrete dynamical systems which for us are numbers N_t were $t = 0, 1, 2, 3, \ldots$ and such that is a function f with

$$N_{t+1} = f(N_t).$$

If N_t is the population size in year t, then the function f lets us compute the population size the next year by the formula above.

In a discrete dynamical system $N_{t+1} = f(N_t)$ the ***equilibrium points*** (also called a ***stationary point*** or a ***rest point***), N_*, are the points solutions to $f(N) = N$. These are of interest because then then if $N_0 = N_*$, then $N_t = N_*$ for all t. That is the population stays the same from year to year.

The equilibrium point N_* is ***stable*** (or ***attracting***) if when N_0 starts close to N_*, then N_t gets closer and closer to N_*.

The equilibrium is ***unstable*** (or ***repelling***) if when N_0 is close to N_*, then N_t moves away from N_*. There is an easy way to check if an equilibrium point is stable or not:

Theorem. *Let N_* be an equilibrium point of $N_{t+1} = f(N_t)$. (That is $f(N_*) = N_*$.) Then*

- *If $|f'(N_*)| < 1$, then N_* is stable.*
- *If $|f'(N_*)| > 1$, then N_* is unstable.*

Problem. A special case of interest is the the ***discrete logistic equation*** with carrying capacity K and per capita grow rate r, that is

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) \tag{1}$$

That is in this case

$$f(N) = N_t + rN_t\left(1 - \frac{N_t}{K}\right)$$

(a) Solve the equation $f(N) = N$ to see that there are two equilibrium points, $N_* = 0$ and $N_* = K$.

(b) Compute $f'(N)$ and show that

$$f(0) = 1 + r$$
$$f(K) = 1 - r.$$

(c) Use this to explain why the following hold

- The equilibrium at $N = 0$ is always unstable.
- The equilibrium at $N = K$ is always unstable is stable when $0 < r < 2$ and unstable when $r > 2$.

(d) Note that the right hand of (1) side of this equation also has a zero when $N = \dfrac{1+r}{r}K$. Does this have any biological meaning. *Answer:* At least to the best of my knowledge the answer is no. □

In class we drew some pictures by taking the initial value N_0, the going up to the graph, then over to the line, then up to the graph and the over again to the line and continuing in this manner. This method is usually called ***cobwebing***. (This is because at least some of the pictures look like a spider's cobweb.) There is a nice YouTube video about these at: https://www.youtube.com/watch?v=nxcKh36rep0.

Here are some graphs you can practice on. The answers are at the end of this file

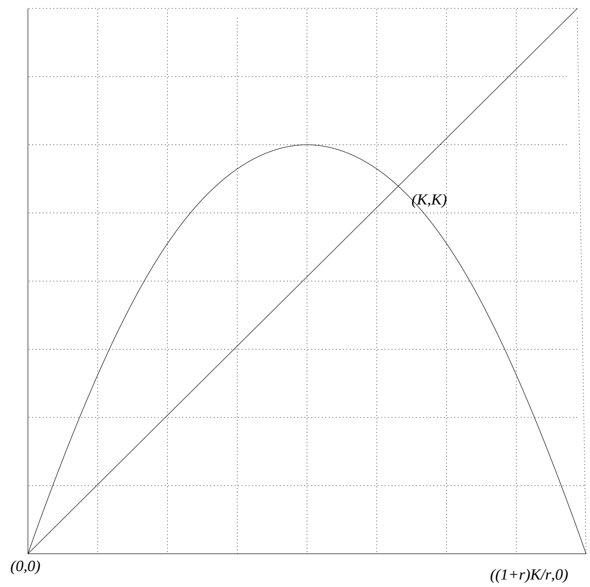

FIGURE 1. The graph of $N_{t+1} = N_t + rN_t\left(1 - \dfrac{N_t}{K}\right)$ as a function of N_t, showing the two equilibrium points $(0,0)$ and (K,K). Also showing that the zeros of the of the right hand side of (1) are $N = 0$ and $N = \dfrac{1+r}{r}K$.

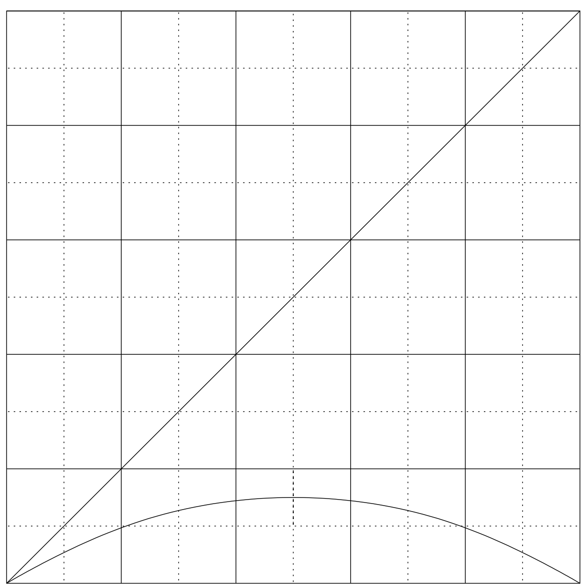

FIGURE 2. In this system there is only one equilibrium point, the one at $N = 0$. Is is stable or unstable? Draw some cobwebs to decide.

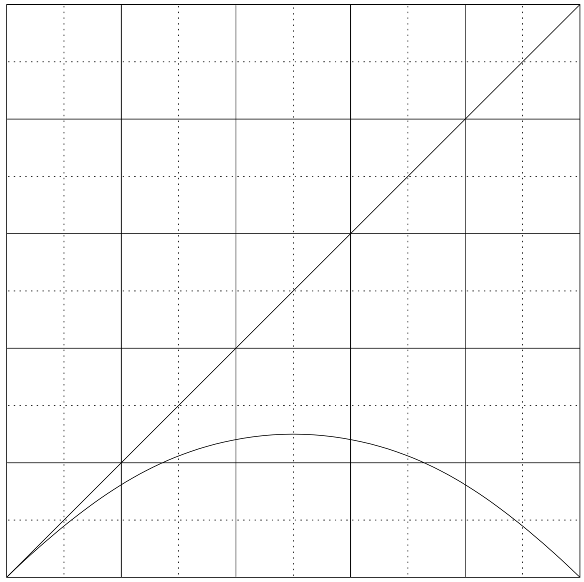

FIGURE 3. Again there is only one equilibrium point, the one at $N = 0$. Is is stable or unstable? Draw some cobwebs to decide.

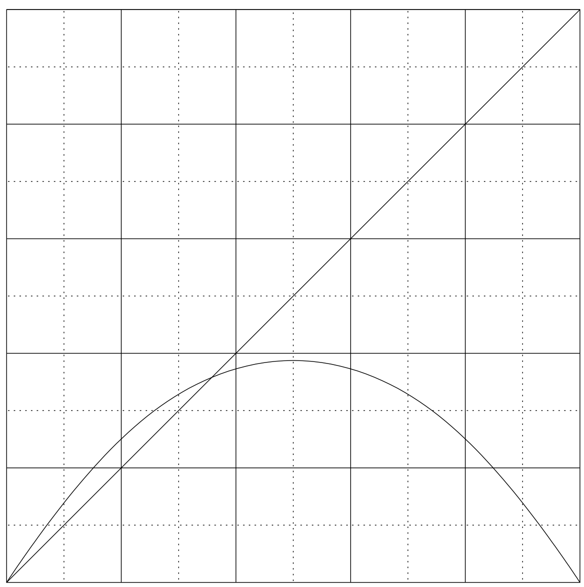

FIGURE 4. Here there are two equilibrium points. Are they stable or unstable?

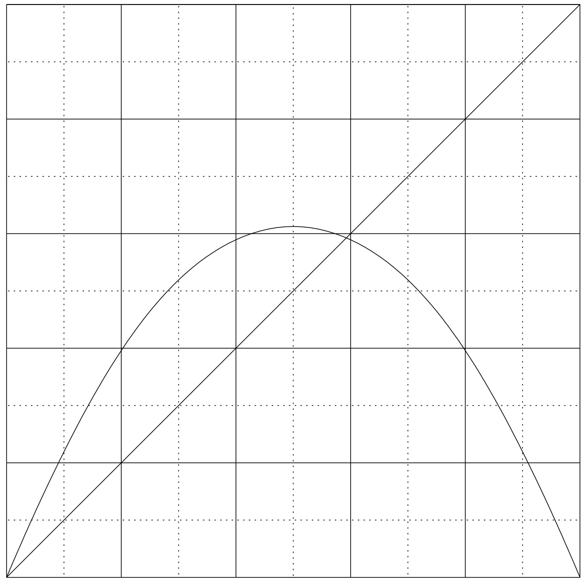

FIGURE 5. Again decide if the two equilibrium points are stable or unstable.

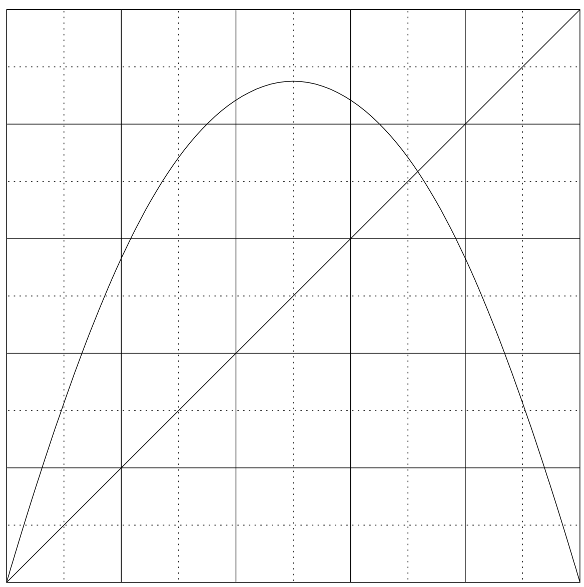

FIGURE 6. Are the two equilibrium points are stable or unstable.

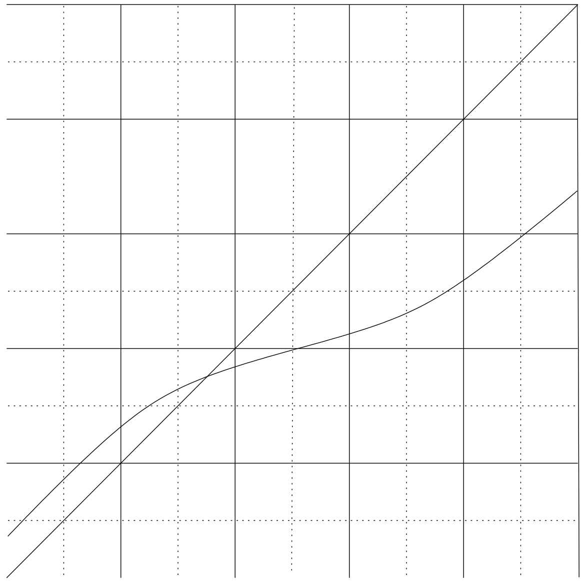

FIGURE 7. Is the equilibrium point are stable or unstable?

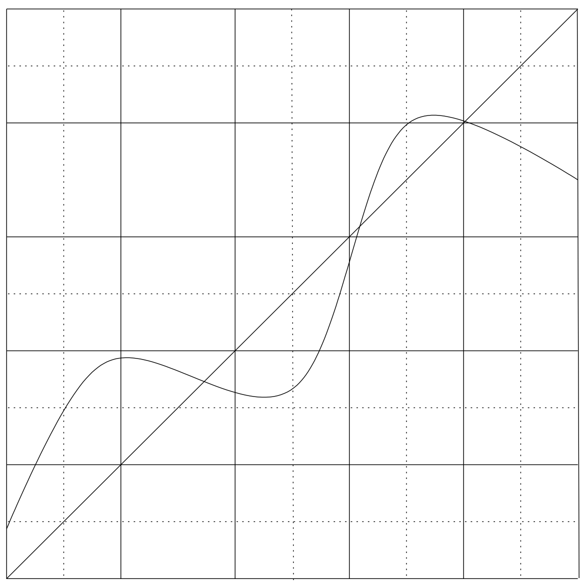

FIGURE 8. Here there are three equilibrium points. Which are stable and which unstable?

Answers:
Figure 2: 0 is stable.
Figure 3: 0 is stable.
Figure 4: 0 is unstable and the other rest point is stable.
Figure 5: 0 is unstable and the other rest point is stable.
Figure 6: Both points are unstable.
Figure 7: The point is stable.
Figure 8: The middle rest point is unstable, the other two are stable.