

Mathematics 172 Homework.

Recall the plot to date on our study of age structured population growth. If we have three stages we let

$$\vec{n}(t) = \begin{bmatrix} n_1(t) & n_2(t) & n_3(t) \end{bmatrix}$$

where $n_j(t)$ is the number of individuals in stage j in year t . We have the **Leslie matrix**

$$L = \begin{bmatrix} f_1 & f_2 & f_3 \\ p_1 & 0 & 0 \\ 0 & p_2 & 0 \end{bmatrix}.$$

In entries have the meanings

f_j = Fecundity of stage j individuals

= average number of offspring to a stage j individual.

p_j = Proportion of stage j individuals that live to stage $j + 1$.

The Leslie matrix tells us how the population changes from one year to the next:

$$\vec{n}(t+1) = L\vec{n}(t).$$

Thus if we know $\vec{n}(0)$ we can compute the numbers in the future by

$$\vec{n}(t) = L^t \vec{n}(0).$$

At least for matrices that are not too large this is easy to do on our calculators.

One of the pieces of information that is **age distribution** in year t . That is the proportion of the population that is in each stage. If

$$\begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}$$

then the total number of individuals is

$$N(t) = n_1(t) + n_2(t) + n_3(t)$$

and the age distribution is given by the vector

$$\frac{1}{N(t)} \vec{n}(t) = \frac{1}{N(t)} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix} = \begin{bmatrix} \frac{n_1(t)}{N(t)} \\ \frac{n_2(t)}{N(t)} \\ \frac{n_3(t)}{N(t)} \end{bmatrix}$$

We computed many examples of this on Quiz 17.

We have a **stable age distribution** if the age distribution stays the same from year to year. That this means is that for years t and $t + 1$ we have

$$\frac{1}{N(t)}\vec{n}(t) = \frac{1}{N(t+1)}\vec{n}(t+1)$$

which can be rewritten as

$$\vec{n}(t+1) = \left(\frac{N(t+1)}{N(t)} \right) \vec{N}(t) = \lambda \vec{N}(t)$$

where

$$\lambda = \frac{N(t+1)}{N(t)}.$$

But we also have

$$\vec{n}(t+1) = L\vec{n}(t).$$

Comparing our two formulas for $\vec{n}(t)$ gives

$$L\vec{N}(t) = \lambda\vec{N}(t).$$

In this case we let

$$\begin{aligned} \lambda &= \textbf{growth ratio} \\ r = \lambda - 1 &= \textbf{per capita growth rate.} \end{aligned}$$

This motivates the following. Let L be a square matrix and \vec{n} a vector and λ a number. Then \vec{n} is an **eigenvector** of L with **eigenvalue** λ if and only if

$$L\vec{n} = \lambda\vec{n}.$$

What this means for us is that $L\vec{n}$ and \vec{n} have the same age distribution. Thus in the context of our class saying that \vec{n} is a eigenvector of L is saying that \vec{n} has the stable age distribution and that the eigenvector λ is the growth ratio.

Let us see what this mean in concrete cases. Let L be the Leslie matrix

$$L = \begin{bmatrix} 0 & 44 & 890 \\ 0.01 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

and let

$$\vec{n} = \begin{bmatrix} 4 \\ 0.02 \\ 0.009 \end{bmatrix}$$

1. Show that

$$L\vec{n} = \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix}$$

□

Also note that

$$2\vec{n} = 2 \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix} = \begin{bmatrix} 2(8) \\ 2(0.04) \\ 2(0.016) \end{bmatrix} = \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix}.$$

Thus \vec{n} is a eigenvector for L with eigenvalue 2. Therefore for this Leslie matrix the stable age distributions is

$$\frac{1}{8 + .02 + .009} \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix} = \begin{bmatrix} 0.99305 \\ 0.00497 \\ 0.00199 \end{bmatrix},$$

the growth factor is $\lambda = 2$ and $r = \lambda - 1 = 1$ is the per capita growth rate.

2. Let L be the Leslie matrix

$$L = \begin{bmatrix} 0 & 3.9 & 15.25 \\ 0.1 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

(a) Show that

$$\vec{n} = \begin{bmatrix} 100 \\ 10 \\ 4 \end{bmatrix}$$

is an eigenvector for L with eigenvalue $\lambda = 1$ and thus that per capita growth rate is $r = \lambda - 1 = 0$.

(b) Find the stable age distribution. *Solution:* It is

$$\begin{bmatrix} 0.87719 \\ 0.08772 \\ 0.03509 \end{bmatrix}$$

(c) If

$$\vec{n}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

find $\vec{n}(1)$, $\vec{n}(2)$, $\vec{n}(5)$ and $\vec{n}(30)$ and the age distribution of these vectors.

Solution:

$$\vec{n}(1) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0 \\ 1.0 \\ 0 \end{bmatrix}$$

$$\vec{n}(2) = \begin{bmatrix} 39 \\ 0 \\ 4 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.90698 \\ 0 \\ 0.09302 \end{bmatrix}$$

$$\vec{n}(5) = \begin{bmatrix} 47.5800 \\ 1.5210 \\ 2.4400 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.92315 \\ 0.02951 \\ 0.04734 \end{bmatrix}$$

$$\vec{n}(30) = \begin{bmatrix} 38.3362 \\ 3.8266 \\ 1.5343 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.87732 \\ 0.08757 \\ 0.03511 \end{bmatrix}$$