Mathematics 172 Homework, October 16, 2019.

We have studied rate equation such as

$$\frac{dy}{dt} = f(y),$$

for example the logistic equation

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$$

where there is single dependent variable, y, depending on time, t. We now wish to study rate equations (also called differential equations) of the form

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

where we have two dependent variables, x and y, depending on time. In some of the examples we will look at x and y will be the population sizes of two species competing for the same resources, or x will be the size of a population of predators and y will be the size of the population of its prey.

We will start by understanding just what such equations say. Let us look at an example.

$$\frac{dx}{dt} = .1x\left(\frac{10 - x - 2y}{10}\right)$$
$$\frac{dy}{dt} = .3y\left(\frac{20 - 3x - y}{20}\right)$$

Note these equations could also be written with the "prime" notation:

$$x'(t) = .1x(t)\left(\frac{10 - x(t) - 2y(t)}{10}\right)$$
$$y'(t) = .3y(t)\left(\frac{20 - 3x(t) - y(t)}{20}\right)$$

1. If $x(3) = 2$ and $y(3) = 1$ what are $x'(3)$ and $y'(3)$?

Solution: Just plug the values for $x(3)$ and $y(3)$ into the equations for $x'(t)$ and $y'(t)$.

$$x'(3) = .1x(3)\left(\frac{10 - x(3) - 2y(3)}{10}\right) = .1 \times 2\left(\frac{10 - 2 - 2 \times 1}{10}\right) = .12$$
$$y'(3) = .3y(3)\left(\frac{20 - 3x(3) - y(3)}{20}\right) = .3 \times 1\left(\frac{20 - 3 \times 2 - 1}{20}\right) = .195$$

2. For the same equations find $x'(6)$ and $y'(6)$ if

(a) $x(4) = 4$, $y(4) = 6$.

Solution: $x'(4) = -.24, \quad y'(4) = .18$

(b) $x(3.1) = 1.5$, $y(3.1) = .4$.

Solution: $x'(3.1) = .1155, \quad y'(3.1) = .0906$

3. Still with the same equations, if $x(0) = 4$ and $y(0) = 6$ (see problem 2a), is $x(t)$ initially increasing or decreasing? Is $y(t)$ initially increasing or decreasing?

Solution: As $x'(0) = -.24$ the derivative of $x(t)$ is initially negative, and a negative derivative implies $x(t)$ is decreasing. Likewise $y'(0) = .18$ so the derivative of y is initially positive and thus $y(t)$ is initially increasing.

Recall our basic approximation formulas:

$$x(t+h) \approx x(t) + x'(t)h$$
$$y(t+h) \approx y(t) + y'(t)h$$

which hold when h is close to zero.

4. For the system

$$\frac{dx}{dt} = .1x + .2y$$
$$\frac{dy}{dt} = -.3x + .4y$$

(a) If $x(2) = 3$ and $y(2) = 5$ approximate $x(2.1)$ and $y(2.1)$. *Solution:* First compute $x'(2)$ and $y'(2)$ to get

$$x'(2) = 1.3$$
$$y'(2) = 1.1$$

Then by the basic approximation formulas

$$x(2.1) \approx x(2) + x'(2)(.1) = 3 + 1.3(.1) = 3.13$$
$$y(2.1) \approx y(2) + y'(2)(.1) = 5 + 1.1(.1) = 5.11$$

(b) If $x(4) = 5$ and $y(4) = 1$ approximate $x(4.05)$ and $y(4.05)$. *Solution:* Use the same method as in part (a) to get

$$x(4.05) \approx 5.035$$
$$y(4.05) \approx 0.945$$