Mathematics 172 Homework.

We have started to look at models for two competing species. Call the two species the x-species and the y-species and let

$$x(t) = \text{size of } x\text{-species population at time } t,$$
$$y(t) = \text{size of } y\text{-species population at time } t,$$

We will assume that the two species are living in the same environment and competing for the same resources.

Let

$$f(x,y) = \frac{1}{x}\frac{dx}{dt} \tag{1}$$

be the relative growth rate of the x-species. The can also be thought of as the per capita growth rate of the x-species. Likewise let

$$g(x,y) = \frac{1}{y}\frac{dy}{dt} \tag{2}$$

be the relative growth rate of the y-species.

- If x and y are both small then there are enough resources to for both of the populations to be unceasing. Thus when both x and y are small both of $f(x,y)$ and $g(x,y)$ are positive.
- If x (or y) is very large, then the x-species (or the y-species) is using up so much of the environment's resources that both the size of the x-species and y-species are decreasing. That is if either x or y is very large then both $f(x,y)$ and $g(x,y)$ are negative.
- If either of x or y is increased, then more of the environment's resources have been used up and so the relative growth rates of both the x-species and the y-species will go down. That is

  $$x_1 < x_2 \text{ and } y_1 < y_2$$

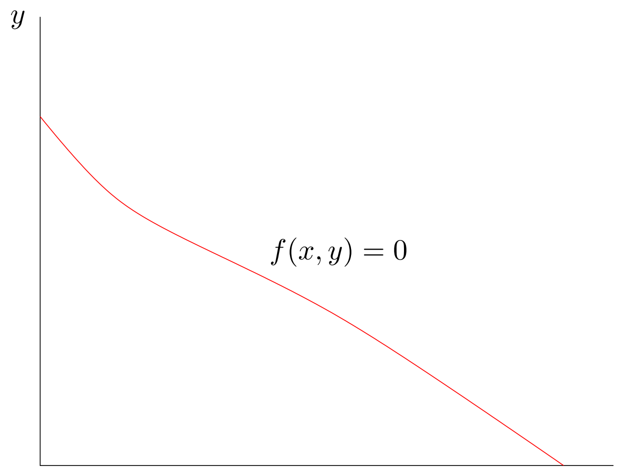
  implies

  $$f(x_1,y_1) < f(x_2,y_2) \text{ and } g(x_1,y_1) < g(x_2,y_2).$$

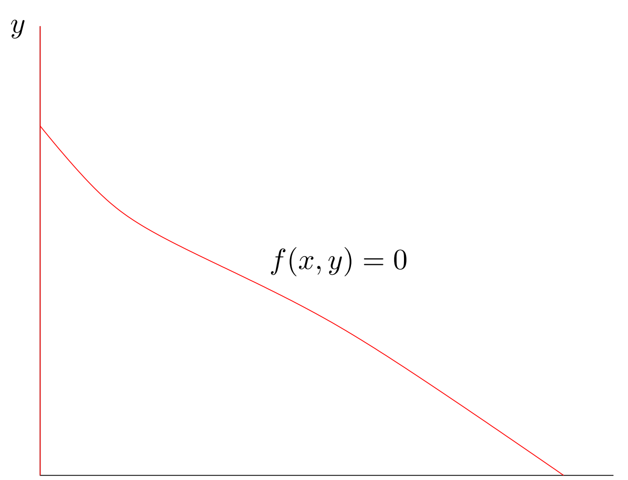
Multiply Equation (1) by x and Equation (2) by y and we get the form of the equation for competing species which we will be working with:

$$\frac{dx}{dt} = xf(x,y)$$
$$\frac{dy}{dt} = yg(x,y).$$

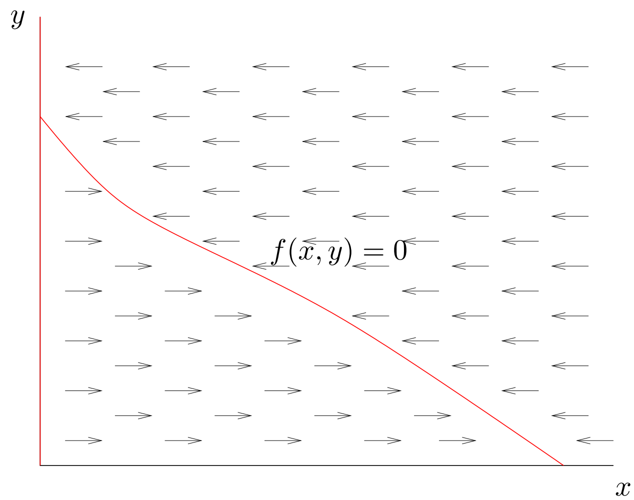
We now analyze when $\frac{dx}{dt} = 0$. This implies that either $x = 0$ (which is the equation of the y-axis) or $f(x,y) = 0$. Our assumptions above imply that the curve $f(x,y) = 0$ will look like the following:



So the complete picture of $\frac{dx}{dt} = 0$ (shown in red) is

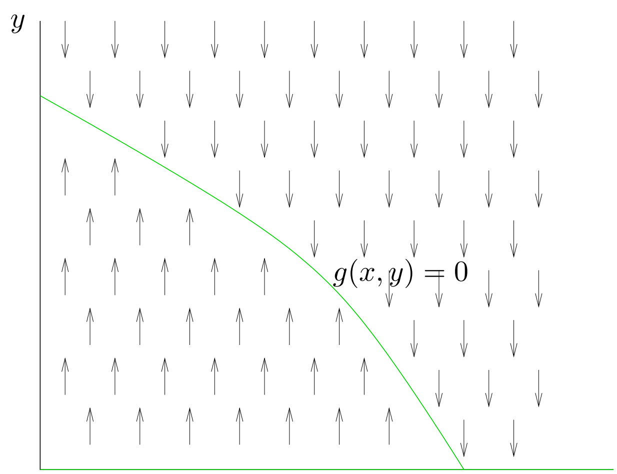


In this figure the points, (x, y), below the curve $f(x, y) = 0$ have $f(x, y) > 0$ and thus $\frac{dx}{dt} > 0$. One way to think of this is that the curve $f(x, y) = 0$ is analogous to the carrying capacity for the x-species and when the population size is below the carrying capacity it is increasing (and thus the relative growth rate $f(x, y)$ is positive). Likewise if (x, y) is above the curve $f(x, y) = 0$, then the population is above the carrying capacity for the x-species and so $\frac{dx}{dt} < 0$. We add arrows to the figure indicating if the x-species is increasing or decreasing.



Problem 1. Do a similar analysis of then $\frac{dy}{dt} = 0$ drawing the set $\frac{dy}{dt} = 0$ in green and put in arrows indicating if y is increasing or decreasing.

Solution:

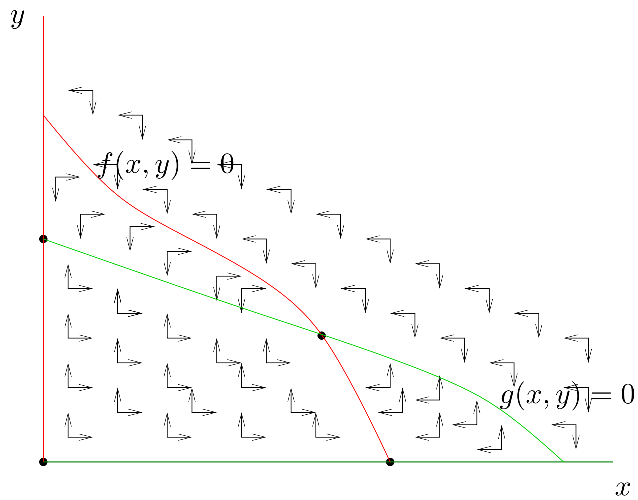


We can now combine the two pictures. Before doing this recall that (x_*, y_*) is an ***equilibrium point*** or ***rest point*** if and only if

$$f(x_*, y_*) = 0 \qquad \text{and} \qquad g(x_*, y_*) = 0.$$

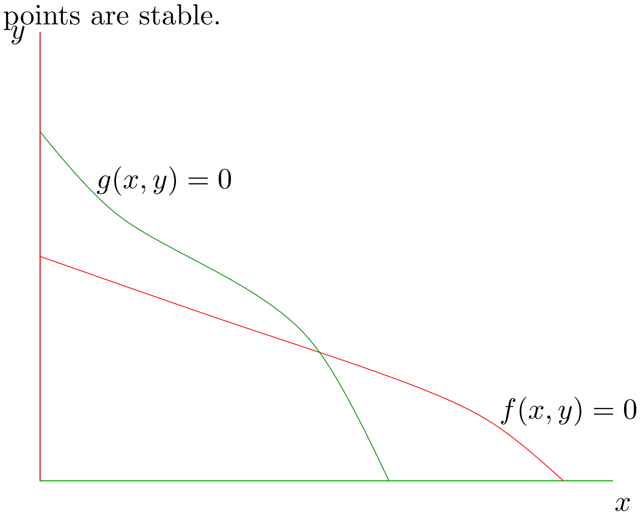
These are the constant solutions.

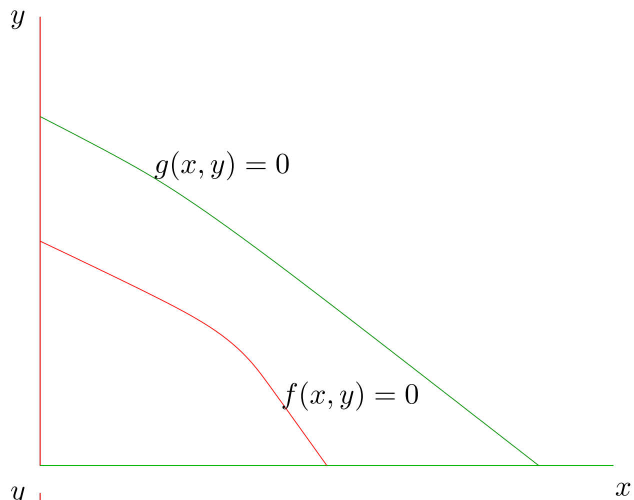
Here is an example of a putting the pictures together:

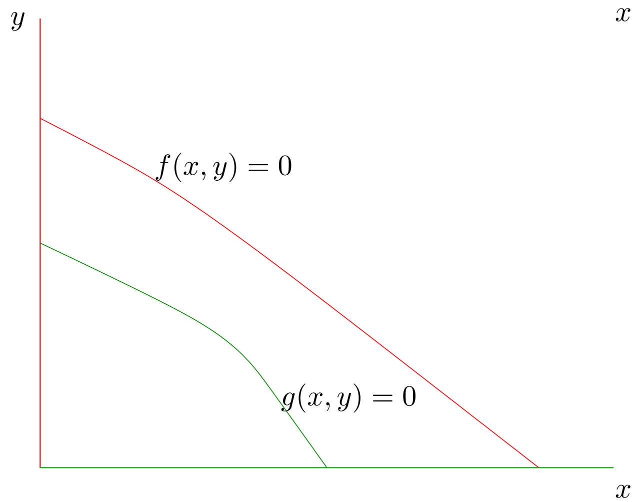


Here the equilibrium points are denoted by the black dots and the arrows indicate which way points are moving. In this case only the central equilibrium point is stable and the other three are unstable.

Problem 2. For the following figures draw in the equilibrium points, arrows showing what direction points are moving and determine which of the equilibrium points are stable.




y
g(x, y) = 0
f(x, y) = 0
x


y
f(x, y) = 0
g(x, y) = 0
x