Mathematics 172 Homework, November 9, 2019.

Our basic model for the spread of disease is the SIR-model. We have a population of organisms of size N with an infection going in it. We split the population into three class:

Susceptible: These are the members of the population that have not been infected and are susceptible to being injected.

Injected: These are the members of the population that are currently infected.

Removed: These are the members of the population that have had the disease and have recovered and are immune, have had the disease and have died, or those that are naturally immune to the disease.

We will use the notation

$$\begin{aligned} S_t &= \text{Number of susceptibles at time } t \\ I_t &= \text{Number of infecteds at time } t \\ R_t &= \text{Number of recovereds at time } t \end{aligned}$$

where we will usually measure time in days.

We will use a discrete model rather than a continuous. This means that instead of using a derivative $\dfrac{dS}{dt}$ we will replace it with

$$\Delta S = S_{t+1} - S_t.$$

The derivative measures the continuous rate of change, while ΔS gives the change from day t to day $t+1$. Since we are counting number of organisms on each day, it makes more sense to work with the day to day change rather than the continuous rate of change.

We will assume that susceptible individual become by exposure to an infected individual and that the likelihood a susceptible becomes infected is proportional to the number of contacts between a it and an infected individual. So we assume there is a constant of proportionality b such that

$$\Delta S = -bSI.$$

That is the number of susceptibles that become infected on a given day, and thus removed from the from the number of susceptibles (whence the minus sign) is proportional to the number of interactions between susceptibles and infecteds.

Thus on a given day the number of new infected individuals will be the number of susceptible individuals that have become infected, which is bSI. We assume the number of individuals removed from the infected population is proportional to the number of infected individuals. On way to think of is that if the duration of an infection is d days, then we expect, roughly,

$k = 1/d$ of the infected population to be removed in a day. Putting these assumption together gives

$$\Delta I = bSI - kI.$$

The change in the removed population is the same as the number of that leaves the infected population. Therefore

$$\Delta R = kI.$$

Putting these equation together:

$$\begin{aligned} \Delta S &= -bSI \\ \Delta I &= bSI - kI \\ \Delta R &= kI \end{aligned}$$

More explicitly

$$\begin{aligned} S_{t+1} - S_t &= -bS_t I_t \\ I_{t+1} - I_t &= bS_t I_t - kI_t \\ R_{t+1} - R_t &= kI_t \end{aligned}$$

which we can farther rearrange as

$$\begin{aligned} S_{t+1} &= S_t - bS_t I_t \\ I_{t+1} &= I_t + bS_t I_t - kI_t \\ R_{t+1} &= R_t + kI_t \end{aligned}$$

We wish to program our TI calculators to deal with this system. As usual the calculator does not like the variables we want to use, so we will use variables it does like, which are $u = S$, $v = I$, and $w = R$. It also likes us to use n rather than t. In calculator friendly notation the SIR equations become

$$\begin{aligned} u(n) &= u(n-1) - bu(n-1)v(n-1) \\ v(n) &= v(n-1) + bu(n-1)v(n-1) - kv(n-1) \\ w(n) &= w(n-1) + kv(n-1). \end{aligned}$$

As a first example let

$$\begin{aligned} b &= .001 \\ k &= .2 \\ u(0) &= S_0 = 990 \\ v(0) &= I_0 = 10 \\ w(0) &= R_0 = 0. \end{aligned}$$

We store the first two of these in the B and K registers. To store the first number press

.001 STO ALPHA ENTER.

(What you will see on the screen is .001→B.) You can check that this has worked by pressin sf 2ND RCL ALPHA B ENTER. Do the same steps to store .2 in the K register.

We now need to set up the calculator to work with tables. To start press the MODE key and the calculator will open up a screen that looks something like this (some of the highlighted boxes may be in different places):

[NORMAL] SCI ENG
[FLOAT] 0 1 2 3 4 5 6 7 8 9
[RADIAN] DEGREE
[FUNC] PAR POL SEQ
[CONNECTED] DOT
[SEQUENTIAL] SIMUL
[REAL] a+bi re^θi
[FULL] HORIZ G-T

Use the cursor key to move down to the forth line and over to SEQ and press enter to change from FUNC mode to SEQ mode. The screen will now look like:

[NORMAL] SCI ENG
[FLOAT] 0 1 2 3 4 5 6 7 8 9
[RADIAN] DEGREE
FUNC PAR POL [SEQ]
[CONNECTED] DOT
[SEQUENTIAL] SIMUL
[REAL] a+bi re^θi
[FULL] HORIZ G-T

Now press 2ND TABLESET and edit until it looks like

TABLE SETUP
TblStart=0
ΔTbl=1
Indpnt : [Auto] Ask
Depend: [Auto] Ask

Now press the Y= key. If you have never used the SEQ mode before it will look like

Plot1 Plot2 Plot2
nMin=
\u(n)=
u(nMin)=
\v(n)=
v(nMin)=
\w(n)=
w(nMin)=

Edit this until it looks what is below. There are some tricks involved in this:

- Where there is an n use the X,T, θ, n key.
- For u, v, and w use 2ND u (over the 7 key), 2ND v (over the 8 key), and 2ND w (over the 9 key).
- For B use press ALPHA B and for K press ALPHA K. (When we run the program the calculator is smart enough to use our stored values of b and k.)

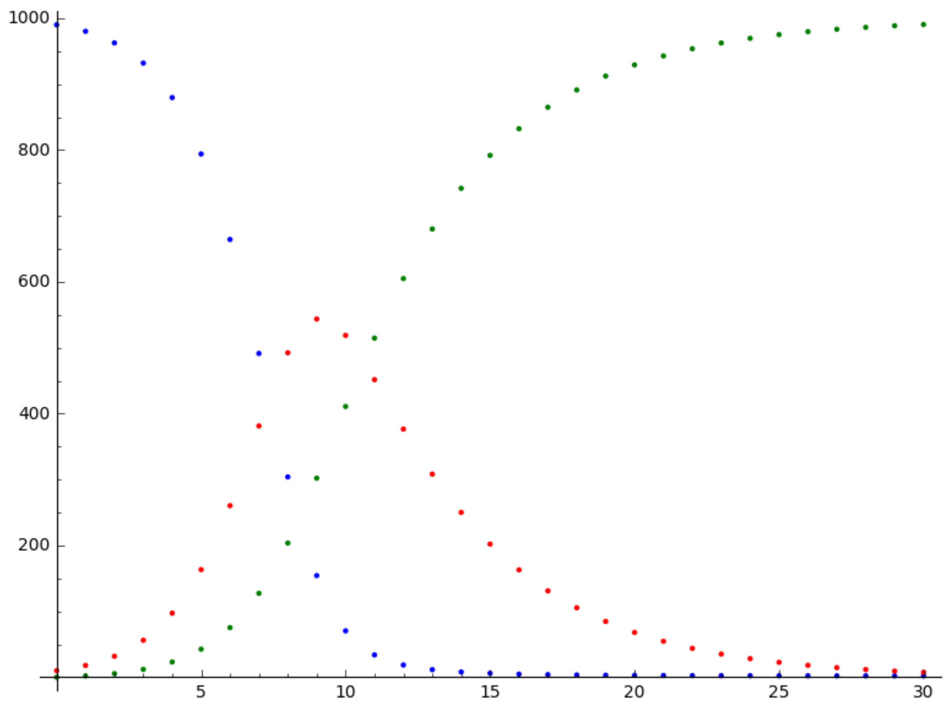
Plot1 Plot2 Plot2
nMin=0
\u(n)=u(n-1)-Bu(n-1)v(n-1)
u(nMin)=990
\v(n)=v(n-1)+Bu(n-1)v(n-1)-Kv(n-1)
v(nMin)=10
\w(n)=w(n-1) + Kv(n-1)
w(nMin)=0

And we are now pretty much done. Press sf 2ND TABLE and you get a table that looks like

n	u(n)	v(n)
0	990	10
1	980.1	17.9
2	962.56	31.864
3	931.89	56.162
4	879.55	97.266
5	794	162.36
6	664.29	260.4

You can get the w values by moving the cursor to the right. Likewise you can scroll down to get the values of u, v and w for values of n larger than 6.

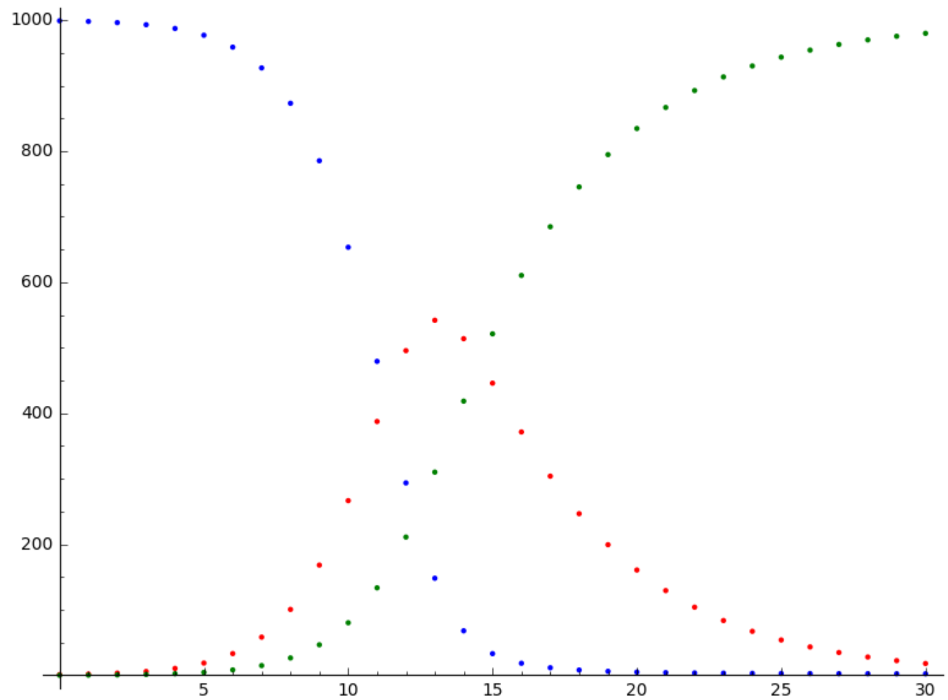
We can also graph this data. Press WINDOW and set
nMin = 0
nMax = 30
Do a ZoomFit and you should get graph that looks like



Now we can ask what happens if we change the initial values to

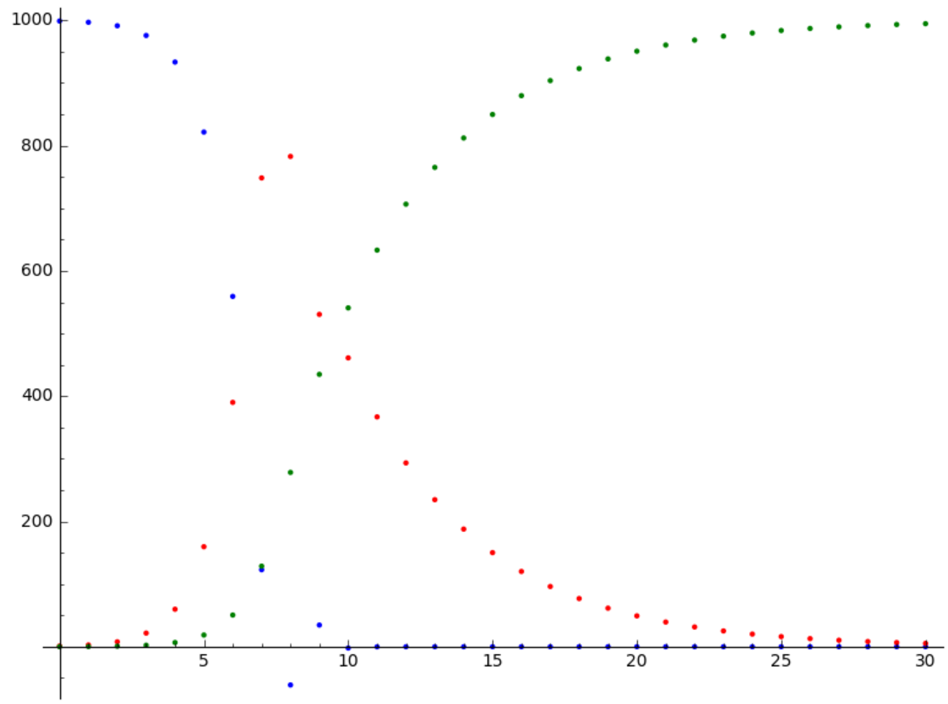
$$S_0 = 999$$
$$I_0 = 1$$
$$R_0 = 0$$

That is we only have one infected to start with. To do this you just go Y= and change the values of u(nMin) and v(nMin) to 999 and 1. The new graph looks like



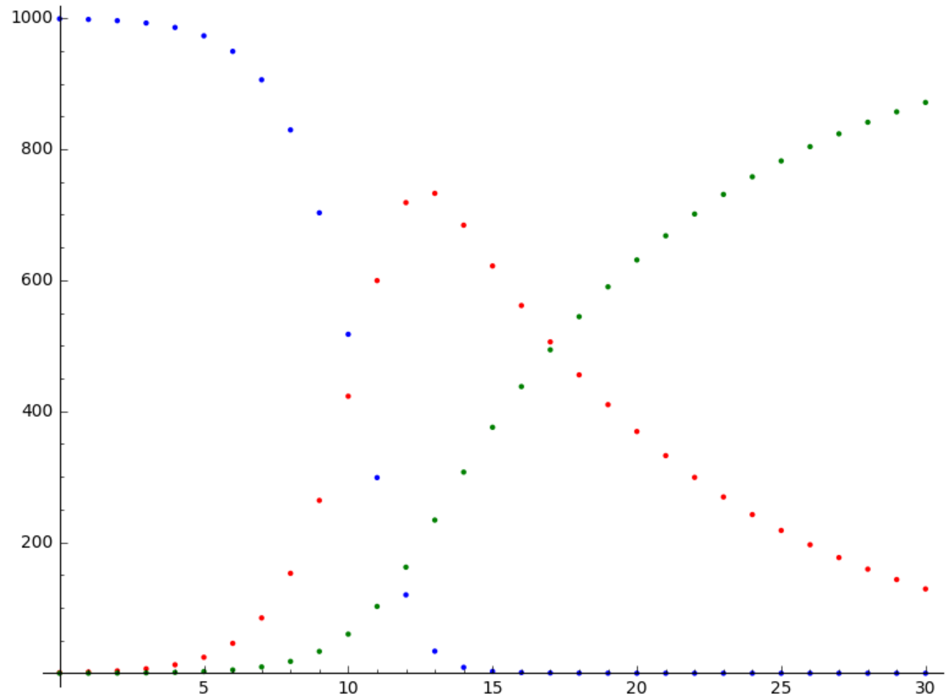
Which is not much of a change. The epidemic is just a little bit slower to get started.

What if we change b to a larger value. This would means that the infection is more contagious. Change the value of b to $b = .002$ by storing .002 in the B register. Still using $S_0 = 999$ and $I_0 = 1$ the graph is



In this case the infection spreads faster (no surprise). The negative values mean that for these values, the model could be improved. But this is still good enough to get an idea of what is going on.

Let us do one more experiment. Change b back to $b = .001$. We have been using the value $k = .2$, which means that the average length of an infection is $1/k = 5$ days. If we change k to $k = .1$, so that the length of an infection is 10 days, then the graph is



Thus lengthening the length of the infection slows how fast it spreads, which I find a bit surprising.